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# Heterogeneity in Nash Networks

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**Abstract**

Heterogeneity in Nash networks can arise due to differences in the following four variables: (i) the value of information held by agents, (ii) the rate at which information decays or loses its value as it traverses the network, (iii) the probability with which a links transmits information, and (iv) the cost of forming a link. In this paper we examine Nash networks, efficient networks and the existence of equilibrium networks under different heterogeneity conditions for the two-way flow model of networks.

*JEL Classification: C72, D85*

*Key Words: Strategic reliability, decay, two-way flow models*

# 1 Introduction

Introducing heterogeneity in network models permits formulations that are more realistic, and therefore can explain reality better. This allows us to address some of the concerns raised about game-theoretic models of network formation. To quote Jackson [7] in this context: *"The weakness of the game-theoretic approach is that most of the explicit characterization of equilibrium networks are so stark that the predicted networks have overly simple structures."* In models which take heterogeneity into account, the predicted networks have possibly complicated architectures. Furthermore, the result in terms of equilibrium networks (and efficient networks) are directly linked with the value of parameters. Thus it also acts as a robustness check for Nash equilibrium and efficiency results obtained using homogeneous parameters.

The notion of Nash networks was introduced by Bala and Goyal in two papers using homogeneous parameters (2000, [1], [2]).[1] In one paper (2000, [1]), links never fail and always transmit all the information reliably. Given that link formation is costly, the authors find that Nash networks are always minimally connected. In their second paper (Bala and Goyal (2000), [2]), each link is allowed to fail with some probability $p$. Although link formation is still a costly act, in this case they find that as the amount of information at stake increases, agents attempt to insure themselves against link failures by forming super-connected networks, i.e, links have back-ups. Subsequent research however shows that the homogeneity of the parameters plays a significant role in these two widely divergent results. Haller and Sarangi (2005, [6]) extend Bala and Goyal (2000, [2]) by allowing different links to have different success probabilities. Interestingly, they show that given

[1] In the following, we use Nash networks to refer to networks that satisfy Nash equilibrium as the stability concept instead of Jackson and Wolinsky's (1996, [8]) notion of pairwise stability.

*any* network $\boldsymbol{g}$, there exists a set of parameter values under which $\boldsymbol{g}$ is Nash – the model with heterogeneity can encompass the results of *both* Bala and Goyal papers.

In this paper we examine different possible heterogeneous Nash network models. In the standard Nash networks model, agents are endowed with some information which can be accessed by other agents forming links with them. Link formation is costly with the cost of establishing a link being incurred by the initiating agent. In these models heterogeneity manifests itself through the payoff function and can occur through four different variables: (i) the value of information held by agents, (ii) the rate at which information decays or loses value as it traverses the network, (iii) the probability with which a link transmits information, and (iv) the cost of forming a link. We focus on the two-way flow models introduced by Bala and Goyal (2000, [1]).[2] The two-way flow model allows bi-directional flow of information through a link regardless of who establishes it. Here we examine Nash networks, efficient networks and the existence of equilibrium networks under different possible heterogeneous frameworks. Our main results concern the existence of Nash networks and the characterization of Nash networks. They can be summarized as follows:

- the introduction of heterogeneity alters the existence of Nash networks, and
- heterogeneity allows all essential networks to be Nash.

The paper is organized as follows. In Section 2, we present the model setup. In Section 3, we exhibit our result concerning models with imperfect reliability. In Section 4, we examine models which incorporate heterogeneity and decay. In Section 5, we discuss our results.

[2]For the role of heterogeneity in one-way flow models see Galeotti (2006, [3]).

# 2 Model Setup

In this section we define the formal elements of the strategic form network formation game. Let $N = \{1, \dots, n\}$, $n \geq 3$, denote the set of players with generic elements $i, j, k$. The set $N$ also constitutes the nodes set. For ordered pairs $(i, j) \in N \times N$, the shorthand notation $i\,j$ is used and for non-ordered pairs $\{i, j\} \subset N$ the shorthand $[i\,j]$ is used.

**Strategies.** We focus only on pure strategies. For player $i$ a pure strategy is a vector $\boldsymbol{g}_i = (g_{i,1}, \dots, g_{i,i-1}, g_{i,i+1}, \dots, g_{i,n}) \in \{0,1\}^{N\setminus\{i\}}$. Since our aim is to model network formation, $g_{i,j} = 1$ is interpreted to mean that there exists a direct link between $i$ and $j$ initiated by player $i$ (link $i\,j$ is formed by $i$), whereas $g_{i,j} = 0$ means that $i$ does not initiate the link ($i\,j$ is not formed). Regardless of what player $i$ does, player $j$ can always choose $g_{j,i} = 1$, i.e., initiate a link with $i$, or set $g_{j,i} = 0$, i.e., not initiate a link with $i$. The set of all pure strategies of agent $i$ is denoted by $\mathcal{G}_i$ and consists of $2^{n-1}$ elements. The joint strategy space is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. Observe further that there is a one-to-one correspondence between the set of joint strategies $\mathcal{G}$ and the set of all directed graphs or networks with vertex set $N$. Namely, to a strategy profile $\boldsymbol{g} = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_n) \in \mathcal{G}$ corresponds the graph $(N, E(\boldsymbol{g}))$ with edges or nodes set $E(\boldsymbol{g}) = \{(i, j) \in N \times N |\ i \neq j,\ g_{i,j} = 1\}$. In the sequel, we shall identify a joint strategy $\boldsymbol{g}$ and the corresponding graph and use the terminology directed graph or directed network $\boldsymbol{g}$.

**Payoffs.** Payoffs of a player $i$, in network formation models are given by the difference between benefits $B_i(\boldsymbol{g})$ and costs $c_i(\boldsymbol{g})$. Hence the payoff of player $i$ in network $\boldsymbol{g}$ is given by

$$u_i(\boldsymbol{g}) = B_i(\boldsymbol{g}) - c_i(\boldsymbol{g}). \tag{1}$$

Next we illustrate different types of heterogeneity in networks by introducing different costs and benefits formulations, starting with link formation costs.

## 2.1 Link Costs

Players incur costs only for the direct links they establish. We consider two possible kinds of costs.

1. Homogeneous costs. Each player $i$ incurs a cost $c > 0$ when she initiates the direct link $i\,j$, i.e., if $g_{i,j} = 1$. Hence the total costs incurred by player $i$ are given by:

$$c_i(\boldsymbol{g}) = c \cdot \sum_{j \neq i} g_{i,j} \tag{2}$$

   when the network $\boldsymbol{g}$ is formed.

2. Heterogeneous costs. The cost of each link depends on the pair of players who form the link. For this case, we write the cost function of player $i$ as follows:

$$c_i(\boldsymbol{g}) = \sum_{j \neq i} g_{i,j} \cdot c_{i,j} \tag{3}$$

## 2.2 Link Benefits

In any given network, benefits depend on values held by agents, the reliability of links and the nature of information decay in the model. Note that reliability and decay models are treated as mutually exclusive scenarios in the networks literature and we retain this distinction. In a model with perfect reliability and no decay, a player obtains the full value of information from all the agents she "observes" both through her direct and indirect links.[3] In a model with imperfect reliability,

[3]We say an agent observes another agent directly if the geodesic distance between them is one, and observes her indirectly if the geodesic distance betwen them exceeds one.

information obtained from a player through a direct link or indirect links has the same value. In other words, information does not diminish in value through indirect links. However, links can fail to transmit information. In this instance it is necessary to take into account all possible paths that link two agents for acquiring the information. This captures the idea that when one channel of communication fails to deliver the information it may be obtainable through another path. In decay models however, links always manage to transmit information but have constraints on how much information they can convey. Since information loses value as it travels along a sequence of links it captures the idea that "it is always better to have it from the horse's mouth." In this scenario instead of considering all possible paths between two agents we only consider the path that conveys the maximum amount of information between two agents. We begin by examining the simplest case.

### 2.2.1 Benefits with Perfectly Reliable Links and no Decay

In such a model, benefits from links depend on the value parameter, $V_{i,j} > 0$. This is the value obtained by agent $i$ from agent $j$. In principal it is also possible for all agents to have the same value of information $V$ which is usually normalized to one. Further, link between agents $i$ and $j$ potentially allows for **two-way flow of information**. The benefits from network $\boldsymbol{g}$ are derived from its closure $\overline{\boldsymbol{g}} \in \mathcal{G}$, defined by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for $i \neq j$. Moreover, since information is acquired through direct and indirect links we say information flows from player $j$ to player $i$, if $i$ and $j$ are linked by means of a path in $\overline{\boldsymbol{g}}$. A **path** of length $m$ in $\boldsymbol{g} \in \mathcal{G}$ from player $i$ to player $j \neq i$, is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct players such that $i_0 = i$, $i_m = j$, and $g_{i_k, i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. Let us denote by

$$N_i(\boldsymbol{g}) = \{j \in N |\ j \neq i, \text{ there exists a path in } \overline{\boldsymbol{g}} \text{ from } i \text{ to } j\},$$

the set of other players whom player $i$ can access or "observe" in the network $\boldsymbol{g}$. Information received from player $j$ is worth $V_{i,j}$ to player $i$. Therefore, player $i$'s benefits from a network $\boldsymbol{g}$ is given by:

$$B_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{i,j}. \tag{4}$$

We now substitute this in (1) and use different cost formulations to obtain a class of models developed by Galeotti, Goyal, and Kamphorst (2006, [4]). Keeping values and costs homogeneous reduces this to one model examined in Bala and Goyal (2000, [1]).

Note that $\overline{\boldsymbol{g}}$ belongs to the set $\mathcal{H} = \{\boldsymbol{h} \in \mathcal{G} | h_{i,j} = h_{j,i} \text{ for } i \neq j\}$. There is a one-to-one correspondence between the elements of $\mathcal{H}$ and the non-directed networks (graphs) with node set $N$. Namely, for $\boldsymbol{h} \in \mathcal{H}$ and $i \neq j$, $[i\ j]$ is an edge of the corresponding non-directed network if and only if $h_{i,j} = h_{j,i} = 1$. In what follows, we identify $\boldsymbol{h}$ with the corresponding non-directed network. Hence, the notation $[i\ j] \in \boldsymbol{h}$ stands for "$[i\ j]$ is an edge of $\boldsymbol{h}$". Also, for $\boldsymbol{k} \in \mathcal{H}$, $\boldsymbol{k} \subset \boldsymbol{h}$ means that $\boldsymbol{k}$ is a subnetwork of $\boldsymbol{h}$.

### 2.2.2 Benefits with Imperfectly Reliable Links

In such a model benefits from links depend on the value parameter, $V_{i,j} > 0$, and the probability of link success, $p_{i,j} > 0$. The closure of the network $\overline{\boldsymbol{g}}$, determines the possible flow of information in this setting. When $\overline{\boldsymbol{g}}_{i,j} = 0$, then there is no (direct) information flow between $i$ and $j$ and if $\overline{\boldsymbol{g}}_{i,j} = 1$, then the link succeeds (there is direct two-way information flow between $i$ and $j$) with probability $p_{i,j} \in (0,1)$ and fails (there is no direct information flow between $i$ and $j$) with probability $1 - p_{i,j}$.

Observe that the joint strategy $\boldsymbol{g}$ gives rise to a random network with possibly different probabilities of realization for different edges. Formally, we treat $\boldsymbol{g}$ and the realizations of the random network as non-directed networks. The possible realizations of the random network consist of the non-directed networks $\boldsymbol{h}$ satisfying $\boldsymbol{h} \subset \overline{\boldsymbol{g}}$. Invoking the independence assumption, the probability of the network $\boldsymbol{h} \subset \overline{\boldsymbol{g}}$ being realized, given $\boldsymbol{g}$ is:

$$\lambda(\boldsymbol{h} \mid \boldsymbol{g}) = \prod_{[i\ j] \in \boldsymbol{h}} p_{i,j} \prod_{[i\ j] \in \overline{\boldsymbol{g}} \setminus \boldsymbol{h}} (1 - p_{i,j}). \tag{5}$$

Note that this conditional probability can also be defined when all links have the same probability of success. Given a strategy profile $\boldsymbol{g}$, player $i$'s expected benefits from the resulting random network is:

$$B_i(\boldsymbol{g}) = \sum_{\boldsymbol{h} \subset \boldsymbol{g}} \lambda(\boldsymbol{h} \mid \boldsymbol{g}) b_i(\boldsymbol{h}). \tag{6}$$

Namely, the realization of the network $\boldsymbol{h}$, which occurs with probability $\lambda(\boldsymbol{h} \mid \boldsymbol{g})$, gives player $i$ benefits $b_i(\boldsymbol{h})$. Summing over all possible realizations $\boldsymbol{h} \subset \boldsymbol{g}$ yields expected benefits. Variations of the expected benefits formulation can be substituted in the payoff function to obtain different models.

The model of imperfect reliability where all values, costs and probabilities are identical across players and links was analyzed by Bala an Goyal (2000, [2]). We call this the **model of imperfect reliability with homogeneous parameters.** The payoff function where the link failure probability is different for each link but $V$ and $c$ are identical across all agents is called the **model of imperfect reliability with heterogeneous links**. Such a model was first analyzed by Haller and Sarangi (2005). On the other hand, a model with identical link failure probabilities, but with heterogeneous values $V_{i,j}$ has been first introduced in this paper. We call this the **model of imperfect reliability with heterogeneous players**. We also allow for heterogeneity in link formation costs.

### 2.2.3 Benefits in Decay Models

Decay models were introduced by Jackson and Wolinksy (1996, [8]) under the name of the "connections model". In the context of Nash networks they were analyzed in Bala and Goyal (2000, [1]). They considered a situation where the value of information, the costs of link formation, and the decay parameter was identical across all agents and links. In other words, they analyzed the case of homogeneous decay. We propose two different frameworks to capture heterogeneity in models with decay.

**Decay with Heterogeneous Players.** Here we utilize the homogeneous decay assumption in conjunction with the heterogeneous players framework of Galeotti, Goyal and Kamphorst (2006, [4]), i.e., we assume that $V_{i,j} \neq V_{k,\ell}$ if $(i,j) \neq (k,\ell)$. Then the payoff function can be written as:

$$u_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \delta^{d_{i,j}(\boldsymbol{g})} V_{i,j} - \sum_{j \in N \setminus \{i\}} c_{i,j} \cdot g_{i,j}, \tag{7}$$

where $\delta$ is the decay parameter and $d_{i,j}(\boldsymbol{g})$ is the distance in the shortest path between $i$ and $j$ in $\boldsymbol{g}$ (or geodesic distance). We retain this name for the model regardless of whether link costs are homogeneous or heterogeneous.

**Decay with Heterogeneous Links.** Here we assume that decay associated with the link $[i\ j]$ is not the same as decay associated with the link $[\ell\ k]$ with $[\ell\ k] \neq [i\ j]$. This assumption captures the fact that on one hand an existing link or chain is not able to transmit all the information through $j$ to $i$ (and vice versa), and on the other hand the quantity of information that a link can convey is not the same across all links. In other words, some links or paths are "better" than others.

We measure the level of decay of a link $[i\ j]$ by the parameter $\delta_{i,j} \in (0,1)$. Given a network $\boldsymbol{g}$, it is assumed that if agent $i$ has formed a link with agent $j$, then agent $i$ receives information of value $\delta_{i,j}$ from $j$. We retain the symmetry assumption, that is $\delta_{i,j} = \delta_{j,i}$. Without loss of generality we assume that the value of a link is $V = 1$. The payoff of agent $i$ in the network $\boldsymbol{g}$ is then given by:

$$u_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \left( \prod_{[\ell\ k] \in C^*_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right) - \sum_{j \in N \setminus \{i\}} c_{i,j} \cdot g_{i,j}, \tag{8}$$

where $C^*_{i,j}(\boldsymbol{g}) = \arg\max_{C_{i,j}(\boldsymbol{g}) \in \mathcal{C}_{i,j}(\boldsymbol{g})} \left\{ \prod_{[\ell\ k] \in C_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right\}$.

It is worth noting that this expression is similar to the expression in Bala and Goyal (2000, [1]) where the decay parameter is homogeneous except that we do not use the geodesic distance anymore.[4] Also as before we retain this name for the model regardless of whether link costs are homogeneous or heterogeneous.

## 2.3 Network Definitions

**Nash Networks.** Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network that remains when all of agent $i$'s links have been removed. Clearly, $\boldsymbol{g} = \boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}$ where the symbol $\oplus$ indicates that $\boldsymbol{g}$ is composed of the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$ (in similar way the symbol $\ominus$ is used to indicate that we remove a link). A strategy $\boldsymbol{g}_i$ is a **best response** of agent $i$ to $\boldsymbol{g}_{-i}$ if

$$u_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq u_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{ for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$

Let $\mathcal{BR}_i(\boldsymbol{g}_{-i})$ denote the set of agent $i$'s best responses to $\boldsymbol{g}_{-i}$. A network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n)$ is said to be a **Nash network** if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each

[4]In Bala and Goyal (2000, [1]), it is assumed that players can always access their own information.

$i \in N$, that is if $\boldsymbol{g}$ is a Nash equilibrium of the strategic game with normal form $(N, (\mathcal{G}_i)_{i \in N}, (u_i)_{i \in N})$. A strict Nash network is a network where all players are playing a strict best response.

**Efficient Networks.** A network $\boldsymbol{g}$ is efficient if the total utility of players is maximum, that is $\sum_{i=1}^{n} u_i(\boldsymbol{g}) \geq \sum_{i=1}^{n} u_i(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{G}$.

**Graph-theoretic Concepts.** We now introduce some definitions of a more graph-theoretic nature. A network $\boldsymbol{g}$ is called a **star** if there is a vertex $i_s$, such that for all $j \neq i_s$, $\max\{g_{i_s,j}, g_{j,i_s}\} = 1$ and for all $k \notin \{i_s, j\}$, $\max\{g_{k,j}, g_{j,k}\} = 0$. Moreover if $g_{i_s,j} = 1$ for all $j \neq i_s$, then the star is an center-sponsored star, otherwise it is a mixed star. A network $\boldsymbol{g}$ is **connected** if there is a path in $\overline{\boldsymbol{g}}$ between all players $i, j \in N$. A network $\boldsymbol{g}$ is **minimally connected** if it is connected and for all $i\,j \in E(\boldsymbol{g})$, $\boldsymbol{g} \ominus i, j$ is unconnected. A network $\boldsymbol{g}$ is **superconnected** if there exists a link $i\,j \in E(\boldsymbol{g})$ such that $\boldsymbol{g} \ominus i, j$ is a connected network.

Finally, we introduce the notion of an essential network. A network $\boldsymbol{g} \in \mathcal{G}$ is **essential** if $g_{i,j} = 1$ implies $g_{j,i} = 0$. Note that if $\boldsymbol{g} \in \mathcal{G}$ is a Nash network, then it must be essential. This follows from the fact that for each link $i\,j$, $c_{i,j} > 0$ and the information flow is two-way and independent of which agent invests in forming the link.

# 3 Models with Imperfect Reliability

Three formulations of network reliability already exist in the literature. First, is the model of Bala and Goyal (2000, [2]) which is homogeneous with respect to all parameters (the $pV$ model). This was followed by the work of Haller and Sarangi

(2005, [6]) who allow for heterogeneity in the links' failure probabilities (the $p_{i,j}V$ model). A third kind of heterogeneity in this framework concerns players with varying values of information. We call this model of imperfect reliability as the model with heterogeneous players (the $pV_{i,j}$ model) and analyze it next.

We begin with an example illustrating that Nash equilibrium under the heterogeneous players model need not coincide with the Nash equilibria under the heterogeneous links model. To allow for meaningful comparison we impose the restriction that the expected value of a link (without transitivity) is the same under both formulations, i.e., $pV_{i,j} = p_{i,j}V$, for all $i, j \in N$.

**Example 1** Suppose $N = \{1, 2, 3\}$, $V = 1$, $p = 1/5$, $c = 1/5$, and

$$\begin{pmatrix} V_{1,1} & V_{1,2} & V_{1,3} \\ V_{1,2} & V_{2,2} & V_{2,3} \\ V_{1,3} & V_{2,3} & V_{3,3} \end{pmatrix} = \begin{pmatrix} 0 & 5 & 9/2 \\ 5 & 0 & 5/3 \\ 9/2 & 5/3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ p_{1,2} & 0 & p_{2,3} \\ p_{1,3} & p_{2,3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3/4 & 9/10 \\ 3/4 & 0 & 1/3 \\ 9/10 & 1/3 & 0 \end{pmatrix}.$$

It is easy to check that $pV_{i,j} = p_{i,j}V$, for all $i, j \in N$. Straightforward calculations show that the network $\boldsymbol{g}$, such that $E(\boldsymbol{g}) = \{1\,3, 3\,2, 2\,1\}$ is a Nash network in the $pV_{i,j}$ framework. But, in the $p_{i,j}V$ framework the difference between the expected utility of player 3 in $\boldsymbol{g}$ and $\boldsymbol{g}'$ with $E(\boldsymbol{g}') = \{1\,3, 2\,1\}$ is $-1/15 < 0$. So, player 3 does not play a best response in $\boldsymbol{g}$ and hence $\boldsymbol{g}$ is not a Nash network. Furthermore, we check easily that $\boldsymbol{g}'$ is a Nash network for the considered $p_{i,j}V$ game.

The intuition behind this diverging results is the following. Since we have $p_{i,j}V = pV_{i,j}$ in the example, the higher $p_{i,j}$ is , the higher is $V_{i,j}$ (for given $V$ and

$p$ respectively). Note that the fact that $V_{1,2}$, $V_{2,3}$ and $p_{1,2}$, $p_{2,3}$ are high relatively to other $V_{i,j}$ and $p_{i,j}$ does not have the same implications for actions. The fact that $V_{1,2}$ and $V_{2,3}$ are high only imply that players have a great incentive to form the links 1 2 and 2 3. The fact that $p_{1,2}$ and $p_{2,3}$ are high, imply that players have a great incentive to form the links 1 2 and 2 3, but also imply the fact that the link 3 1 is less necessary for player 3 (if the links 1 2 and 2 3 exist). Indeed, in such a case, this link increases a few the expected payoff of 3. In other words, in the $p_{i,j}V$ framework the values of $p_{1,2}$, $p_{2,3}$ have consequences on the incentive of player 3 to form the link 3 1, unlike in the $pV_{i,j}$ framework. Indeed, in the latter case, the values of $V_{1,2}$, $V_{2,3}$ have no consequence on the incentive of player 3 to form links.

We present two results regarding this model. First, instead of explicitly characterizing all the Nash equilibria, for the sake of brevity, we demonstrate that there exist conditions such that every essential network can be strict Nash. This is also shown to be true for efficient networks. Second, we show by means of an example that Nash networks may not always exist.

**Theorem 1** *Imperfect reliability with heterogeneous players. Let $\boldsymbol{g}$ be an essential network. Then, there exist a homogeneous link cost $c > 0$, a probability $p$, and an array $\boldsymbol{V} = [V_{i,j}]$ of values such that*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*
2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game.*

**Proof** Let $\boldsymbol{g}$ be an essential network.

1. Firstly, we prove that there is a link cost $c > 0$, a probability $p$, and an array $\boldsymbol{V}$ of values such that $\boldsymbol{g}$ is a strict Nash network. We assume that $V_{i,j} \in \{\varepsilon, V\}$. More precisely, if $g_{i,j} = 1$, then $V_{i,j} = V$, and if $g_{i,j} = 0$, then $V_{i,j} = \varepsilon < V$.

Let $\boldsymbol{g}' \oplus i, j$, with $V_{i,j} = \varepsilon$, be the network where $i$ obtains the largest amount of resources thanks to the link $i\,j$. Then, the amount of resources that player $i$ obtains through $j$ is given by:

$$\varepsilon p + \sum_{\ell=2}^{n(n-1)/2} p^\ell (a_\ell \varepsilon + a'_\ell V) \tag{9}$$

where $a_\ell, a'_\ell \in \mathbb{Z}$ for each $\ell \in \{1, \ldots, n(n-1)/2\}$, it is possible that some $a_\ell$ or some $a'_\ell$ are null. Let $A = \sum_{\ell=2}^{n(n-1)/2} |a_\ell| + |a'_\ell|$.

Let $\boldsymbol{g}' \ominus i, j$ be the network where $i$ obtains with the largest probability the resources of $j$, with $V_{i,j} = V$, given that there is no link between $i$ and $j$ in this network. Then, the amount of ressources of $j$ that player $i$ obtains, in $\boldsymbol{g} \ominus i, j$, is given by:

$$\sum_{\ell=2}^{(n-1)(n-2)/2} p^\ell b_\ell V$$

where $b_\ell \in \mathbb{Z}$ for each $\ell \in \{1, \ldots, (n-1)(n-2)/2\}$, it is possible that $b_\ell$ is null. Let $B = \sum_{\ell=2}^{(n-1)(n-2)/2} |b_\ell|$.

Given $A$ and $B$, we there exists $p$ such that

$$p < \frac{V - \varepsilon}{V(A+B)},\ A \neq 0 \text{ or } B \neq 0,$$

and

$$p\varepsilon + p^2 AV < c < Vp - Bp^2 V.$$

Now, it is clear that a player $i$ who did not form a link with $j$ in $\boldsymbol{g}$ has no incentive to form this link. Indeed, if $i$ forms a link with $j$, then her payoff function increases by an amount which is bounded above by $p\varepsilon + p^2 AV - c < 0$. Likewise, a player $i$ has an incentive to preserve her link with player $j$ if $g_{i,j} = 1$. Indeed, a player $j$ brings to $i$ an amount which is bounded below by $Vp - Bp^2 V - c$. So, by assumption player $i$ has no incentive to delete the link $i \quad j$. It follows that all players play a best response in $\boldsymbol{g}$, and $\boldsymbol{g}$ is a strict Nash network.

2. Secondly, we prove that there is a link cost $c > 0$, a probability $p$, and an array $\boldsymbol{V}$ of values such that $\boldsymbol{g}$ is an efficient network. We assume that if $g_{i,j} = 1$, then $V_{i,j} = V$, and if $g_{i,j} = 0$, then $V_{i,j} = \varepsilon < V/2$. Let $\boldsymbol{g}' \oplus i, j$ be the network where the $n$ players obtain the largest amount of resources thanks to the link $i\, j$. Then, the total amount of resources that all players obtain thanks to the link $i\, j$ is given by:

$$2\varepsilon p + \sum_{\ell=2}^{(n-1)(n-2)/2} p^{\ell}(d_{\ell}\epsilon + d'_{\ell}V) \tag{10}$$

where $d_{\ell}, d'_{\ell} \in \mathbb{Z}$ for each $\ell \in \{1, \ldots, n(n-1)/2\}$, it is possible that some $d_{\ell}$ or some $d'_{\ell}$ are null. Let $D = \sum_{\ell=2}^{n(n-1)/2} |d_{\ell}| + |d'_{\ell}|$.
We can find $p$ such that

$$p < \frac{V - 2\varepsilon}{(D+B)V},\ D \neq 0 \text{ or } B \neq 0,$$

and

$$2p\varepsilon + p^2 DV < c < Vp - Bp^2 V$$

Then, by using similar arguments as in the previous part, we conclude that $\boldsymbol{g}$ is an efficient network.

□

First, although Haller and Sarangi (2005, [6]) do not investigate this issue it can be easily shown that a similar efficiency result holds for the heterogeneous links model. Second, Theorem 2 holds if there is heterogeneity in link costs. A simple continuity argument can be used to verify this. Finally, one can ask the question: Given theorem 1 is there any role for explicit characterization of the equilibrium networks? Although for the sake of brevity we do not pursue this, it can still be a meaningful exercise since the framework allows for the existence of multiple equilibria. Consider for instance Example 1 in Bala and Goyal (2000, [2]). This is the simplest possible scenario since it only has 3 players with homogeneous values costs and reliability. First, there exists a parameter range in which periphery-sponsored stars are Nash. Clearly, there is no rule for selecting the central agent, a coordination problem that may still persist under heterogeneity. Moreover, they identify a parameter range where two kinds of architectures are Nash: the center-sponsored star and mixed stars. Such an outcome is possible even with parameter heterogeneity. We now identify sufficient conditions for the existence of star networks in the heterogeneous players framework. Let $V^{\overline{m}} = \max_{(i,j) \in N \times N} \{V_{i,j}\}$ and $V^{\underline{m}} = \min_{(i,j) \in N \times N} \{V_{i,j}\}$.

**Proposition 1** *Let the payoff be given by (1) and let the costs be homogeneous. If $pV^{\underline{m}} > c$ and $(n-2)(1-p^2)V^{\overline{m}} + V^{\overline{m}}(1-p) < c$, then the center-sponsored stars and the mixed stars are strict Nash networks.*

**Proof** The proof by contradiction is an adaptation of the proof of Bala and Goyal (2000, [2]). Let $i_s \in N$ be the central agent of the star. We focus on center-sponsored stars. Suppose that there is an agent $j$ who is not linked with $i_s$ ($g_{j,i_s} = g_{i_s,j} = 0$). Then the marginal payoff that $i_s$ obtains from a player $j$ is $pV_{i_s,j} \geq pV^{\underline{m}} > c$, a contradiction. Moreover, if player $j$ forms $k \geq 1$ links, then her payoff is bounded above by

$$\sum_{i \in N \setminus \{j\}} V_{j,i} - kc. \tag{11}$$

Furthermore, the payoff that player $j$ obtains in the center-sponsored star is:

$$pV_{j,i_s} + p^2 \sum_{i \in N \setminus \{j, i_s\}} V_{j,i}. \tag{12}$$

Subtracting (12) from (11) shows that $j$'s maximum incremental payoff from one or more links is no larger than $(n-2)(1-p^2)V^{\overline{m}}+V^{\overline{m}}(1-p)-c$, which is negative by choice of $p$ and $c$. Hence, $j$'s best response is to form no link with a player $i \neq i_s$. The proof for the mixed stars is made with similar arguments. $\square$

**Existence of Nash networks.** We now show that there exist parameters $p$, $c$, $(V_{i,j})_{(i,j)\in N\times N\backslash\{i\}}$, such that there is no Nash network in the $pV_{i,j}$ framework. This result is preserved when the costs of link formation are heterogeneous.

**Example 2** *Non-existence of Nash networks.* Let $pV_{i,j} < c$ for all links $i\ j$ except the link 1 3. Also, let $pV_{21}+p^2V_{23} > \text{Max}\left\{0, p^2V_{2,1}+pV_{2,3}\right\}$, $p(1-p)(pV_{1,2}+(1+p)V_{1,3}) < c$, $p(1-p)(pV_{2,1}+(1+p)V_{2,3}) < c$, $p(1-p)(pV_{3,1}+(1+p)V_{3,2,}) > c$.

1. The empty network is not a Nash network, since $pV_{1,3} > c$.
2. A network with one link cannot be a Nash network. Indeed, if this is the link $i\ j \neq 1\ 3$, then player $i$ can obtain a higher payoff by deleting this link, since $pV_{i,j} < c$ . If this is the link $i\ j = 1\ 3$, then player 2 has a higher payoff if she forms the link 2 1, since $pV_{2,1}+p^2V_{2,3} > 0$.
3. Next, a network $\mathbf{g}$ with two links cannot be a Nash network. Given that Nash networks must be essential in such a network, there always exists a path in $\overline{\mathbf{g}}$ between the players.
   i. Since $pV_{i,j} < c$ for all links $i\ j$, except the link 1 3, no network where a link allows access to resources of only one other player can be Nash (except for the link 1 3). From this it follows that only networks with links $\{1\ 3, 2\ 3\}$, $\{1\ 2, 3\ 2\}$, $\{2\ 1, 3\ 1\}$ or $\{1\ 3, 2\ 1\}$ can be Nash.
   ii. We know that player 2 prefers the link 2 1 to the link 2 3 (since $pV_{2,1}+p^2V_{2,3} > p^2V_{2,1}+pV_{2,3}$). Thus, networks with links 1 3, 2 3, cannot be Nash.

iii. We know that, every thing equal, player 1 prefers the link 1 3 to the link 1 2 (since $pV_{1,3} > c > pV_{1,2}$). Thus, network with links 1 2 and 3 2 cannot be Nash.

iv. Since $p(1-p)(pV_{3,1} + (1-p)V_{3,2}) > c$, a two links network that does not contain the direct link between players 3 and 1, cannot be a Nash network. Thus, neither network with the links 2 1, 3 1 nor the network with the links 1 3, 2 1 can be Nash networks.

Hence, it follows that a network with two links can not be a Nash network.

4. Finally,we show that a network **g** with three links cannot be a Nash network. In this network there always exist a path in $\overline{\mathbf{g}}$ between the players.

   i. Since $p(1-p)(pV_{1,2} + (1+p)V_{1,3}) < c$, no network where player 1 has formed links can be a Nash network. Thus, only networks with the links $\{2\ 3, 3\ 1\}$, $\{2\ 1, 3\ 1\}$ can be Nash.

   ii. Since $p(1-p)(pV_{2,1} + (1+p)V_{2,3}) < c$, no network where player 2 has formed links can be a Nash network. Then, the networks with the following set of links $\{2\ 3, 3\ 1\}$, $\{2\ 1, 3\ 1\}$ can not be Nash.

   It follows that no network with three links can be a Nash network.

# 4 Models with Decay

In this section we focus on situations where links do not convey the full of information. Bala and Goyal (2000, [1]) analyze such networks in a homogeneous setting. Their main result consists in providing some conditions which allow some architectures (complete and empty networks, stars) to be Nash. They argue that it is difficult to provide a complete characterization of Nash or efficient networks in the presence of decay. We begin with situations where the parameters of decay are homogeneous and the values of players are heterogeneous.

## 4.1 Decay with Heterogeneous Agents

In this section we obtain two main results.[5] First, we demonstrate that all networks can be supported as strict Nash and efficient. Second, we show that there exist some situations where there does not exist any Nash network.

**Theorem 2** *Decay with heterogeneous agents. Let $\boldsymbol{g}$ be an essential network. If the payoff function satisfies equation (7), then there exist a link cost $c > 0$ and an array $\boldsymbol{V} = [V_{i,j}]$ of values such that:*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*

2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game.*

**Proof** Let $\boldsymbol{g}$ be an essential network and $V^{\overline{m}} = \max_{(i,j)\in N^2}\{V_{i,j}\}$. We successively prove the two parts of the theorem.

1. For $g_{i,j} = 1$, let $V_{i,j}(1-\delta) > c$, and if $g_{i,j'} = 0$, then let $c > V_{i,j'} + (n-2)\delta V^{\overline{m}}$. These two conditions are compatible if $\delta$ is sufficiently close to zero. Now, it is obvious that player $i$ has no incentive to form a link with $j$ if $g_{i,j} = 0$, since the costs of establishing this link is greater than the maximum value that player $j$ can provide to player $i$. Likewise, it is straightforward that if $g_{i,j} = 1$ player $i$ has no incentive to delete the link with $j$, since $i$ cannot obtain more than $\delta V_{i,j}$ if she accesses to the resources of $j$ through an indirect link.

2. For $g_{i,j} = 1$, let $V_{i,j}(1-\delta) > c$, and if $g_{i,j} = 0$, then let $c > (n-2)V_{i,j} + (n-2)^2 \left(\delta \sum_{k\in N_i(\boldsymbol{g})} V_{k,j}\right)$. These two conditions are compatible if $\delta$ is sufficiently

[5] Galeotti, Goyal and Kamphorst (2006, [4]) consider the case of small levels of decay with heterogeneous agents in the context of the insider-outsider model. However, their results are limited since they only consider two groups of players.

close to zero. We can show that $\boldsymbol{g}$ is an efficient network by using the same arguments as in the previous part.

□

Note if we assume that decay begins with indirect neighbors, then we can construct the following example where some essential networks are neither Nash nor efficient, regardless of the value of the parameters.

**Example 3** Let $N = \{1, 2, 3, 4\}$ be the set of players, let $\boldsymbol{g}$ be a network such that $E(\boldsymbol{g}) = \{1\,2\}$. Then, $\boldsymbol{g}$ is not a Nash network. Indeed, if player 1 has an incentive to form a link with player 2, then $V < c$. In that case, player 3 has an incentive to form a link with player 1. Likewise $\boldsymbol{g}$ is not an efficient network.

*Existence of Nash networks.* We begin by showing that the model of Bala and Goyal (2000, [2]), always has a Nash network. This result continues to hold if heterogeneity is not "too high", more precisely if $V_{i,j} = V_i$ for all $i \in N$.

**Theorem 3** *Suppose costs of setting links are homogeneous.*

1. *If the decay parameter and the values are homogeneous, then there always exists a Nash network.*
2. *If the payoff function satisfies equation (7) and, for all $i \in N$, $V_i = V_{i,j}$, for all $j \in N \setminus \{i\}$, then there always exists a Nash network.*

**Proof** We prove successively the two parts of the proposition.

1. If $\delta V \leq c$, then the empty network is a Nash network. If $\delta V > c$, then there are two possibilities: (i) if $(\delta - \delta^2)V \leq c$, then star networks are Nash, and (ii) if $(\delta - \delta^2)V > c$, then the complete network is Nash.
2. If $\delta V_i \leq c$ for all $i \in N$, then the empty network is a Nash network. If there exist players $i \in N$ such that $\delta V_i > c$, then we start with the empty network

and we let one of these players, say $i_0$, form link with all other players. Then we obtain an center-sponsored star. If for all $i \in N \setminus \{i_0\}$, $(\delta - \delta^2)V_i \leq c$, then the previous network is Nash, otherwise, we let all players $i \in N \setminus \{i_0\}$ such that $(\delta - \delta^2)V_i > c$ be connected to all other players, and we obtain a Nash network.

□

Notice that the above proof also provides conditions under which we can obtain some simple architectures like stars as Nash networks. Our next result shows that the above result is not true for the decay model with heterogeneous players and identical costs of link formation. Note that it is also possible to modify the next example to incorporate heterogeneity in costs of link formation and still achieve the same outcome.

**Example 4** *Non-existence of Nash networks.* Let $N = \{1, \ldots, 5\}$ be the population of players, and assume that:

1. $V_{1,2}\left(\delta - \delta^4\right) + V_{1,3}(\delta^2 - \delta^3) > c$, $\delta V_{1,3} < \delta V_{1,2} < c$, and for all $j \neq 2$, $\delta V_{1,j} + \delta^2 \sum_{k \neq j} V_{1,k} < c$.
2. $V_{2,3}\left(\delta - \delta^4\right) + V_{2,4}(\delta^2 - \delta^3) < c$, $\delta V_{2,3} + \delta^2 V_{2,4} + \delta^3 V_{2,5} + \delta^4 V_{2,1} > c$, and for all $j \neq 3$, $\delta V_{2,j} + \delta^2 \sum_{k \neq j} V_{1,k} < c$.
3. $\delta V_{3,4} > c$ and $\delta \sum_{k \neq 4} V_{3,k} + \delta^2 V_{3,4} < c$.
4. $\delta V_{4,5} > c$ and $\delta \sum_{k \neq 5} V_{4,k} + \delta^2 V_{4,5} < c$.
5. $\delta V_{5,1} > c$ and $\delta \sum_{k \neq 1} V_{5,k} + \delta^2 V_{5,1} < c$.

These five points just list the players with whom each of the others has no incentives to form links as well as those who with whom they would like to form links. For example, the first point implies that player 1 will never form a link with players 3, 4 and 5. From all of this it follows that a Nash network must contain the links 3 4, 4 5, 5 1. It follows that there is four possible Nash networks: $E(\boldsymbol{g}^1) = (3\,4, 4\,5, 5\,1, 1\,2, 2\,3)$, $E(\boldsymbol{g}^2) = (3\,4, 4\,5, 5\,1, 1\,2)$ $E(\boldsymbol{g}^3) = (3\,4, 4\,5, 5\,1)$, $E(\boldsymbol{g}^4) = (3\,4, 4\,5, 5\,1, 2\,3)$. We know by point 2. that player 2 prefers the network $\boldsymbol{g}^2$ to the network $\boldsymbol{g}^1$, so $\boldsymbol{g}^1$ is not Nash. Likewise, player 1 prefers the network $\boldsymbol{g}^3$ to the network $\boldsymbol{g}^2$ by point 1, so $\boldsymbol{g}^2$ is not Nash. Player 2 prefers $\boldsymbol{g}^4$ to $\boldsymbol{g}^3$ by point 2, so $\boldsymbol{g}^3$ is not Nash. Lastly, by point 1 player 1 prefers the network $\boldsymbol{g}^1$ to the network $\boldsymbol{g}^4$, so $\boldsymbol{g}^4$ is not Nash.

## 4.2 Decay with Heterogeneous Links

In this section we consider situations where players have homogeneous values while the decay of each link is different. We obtain the following results.

**Theorem 4** *Let $\boldsymbol{g}$ be an essential network. If the payoff function satisfies equation (8) and costs of setting links are homogeneous, then there exist $c > 0$ and an array $\boldsymbol{\delta} = [\delta_{i,j}]$ of decay such that:*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*
2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game.*

**Proof** We successively prove the two parts of the proposition.

1. Let $\boldsymbol{g}$ be an essential network. For $g_{i,j} = 1$, let $c < \left(\delta_{i,j} - (\delta^m)^2\right) V$, with $\delta^m = \max_{(i',j') \in N^2}\{\delta_{i',j'}\}$. Also, for $g_{i,j} = 0$, let $c > (\delta_{i,j} + \delta^m \delta_{i,j}(n-2))V$. It can be checked that these two conditions are compatible. We note that under these conditions a player $i$, who has not formed a link with player $j$, has no incentive to form a link with her. Indeed, the situation where player $i$ has the greatest incentive to delete a link with $j$ occurs when she obtains the resources of $j$ from a player $k$ such that $\max\{g_{k,i}, g_{i,k}\} = 1$. The condition which allows player $i$ to maintain her link with $j$ is: $c < (\delta_{i,j} - \delta_{i,k}\delta_{j,k})\, V$.

This condition is always true if $c < \left(\delta_{i,j} - (\delta^m)^2\right) V$. Likewise, a player $i$ who has not formed a link with a player $j$ has no incentive to form a link with her. Indeed, a player $j$ can provide at most information of value $(\delta_{i,j} + \delta^m \delta_{i,j}(n-2))V$ to player $i$.

2. The proof of the second part of the proposition is similar to the previous part, but now we assume that if $g_{i,j} = 0$, then $c > (n-2)(\delta_{i,j} + \delta^m \delta_{i,j}(n-2))V$. Since the two conditions are again compatible, we can conclude.

□

**Existence of Nash equilibrium.** The question of existence of Nash equilibria in the models with decay and heterogeneous links remains an open question. However, if we add the heterogeneity of costs to the heterogeneity of links, then we can adapt an argument from Haller, Kamphorst and Sarangi (2006, [5]) to show that Nash equilibria do not always exist. Indeed, they show in example 2 that there exist situations with $\delta = 1$ where links are perfectly reliable, values are homogeneous and costs are heterogeneous, such that there does not exist any Nash network. Hence, by continuity it is possible to construct a similar example with $\delta$ sufficiently close to 1 where Nash equilibria will not exist.

# 5 Discussion

We now sum up the main insights that we obtain with the introduction of heterogeneity. The table 1 provides an overview of the results. The first column indicates the scope of strict Nash networks and the second column does the same for efficient networks. The third column indicates whether existence of Nash networks is always guaranteed.

In going from a deterministic model with homogeneous parameters to a homogeneous probabilistic link failure model, Bala and Goyal (2000 [1], [2]) find that Nash networks change from being minimally connected to super-connected. More precisely, they find that the strict Nash networks change from being empty and center-sponsored stars to being empty and connected networks. The same happens when we allow for decay with homogeneous parameters. At the same time in the Galeotti, Goyal and Kamphorst (2006, [4]) formulation, with heterogeneity in values and costs in the deterministic framework, empty and minimal networks with center-sponsored stars can be supported as Nash. Moreover, the authors show that only minimal networks can be Nash. In contrast, heterogeneity in imperfect reliability models, whether it is of the heterogeneous links type "à la" Haller and Sarangi (2005, [6]), or heterogeneous player type as shown in this paper, always yields an "anything goes" result implying any network can be sustained as strict Nash by an appropriate set of parameters. We find that the same analysis holds when we introduce heterogeneity in decay models. Moreover, Table 1 also shows that a similar trend holds for efficiency. The key insight here is that when heterogeneity requires agents to take into account alternative paths between agents instead of just affecting values and costs of link formation, then it is possible to obtain a richer set of networks.

Existence of equilibrium in the different models is a more tricky issue. Bala and Goyal (2000, [1]) show the existence of equilibrium in the homogeneous deterministic framework through a constructive proof. Haller, Kamphorst and Sarangi (2005, [5]) show that in the deterministic setting there always exists a Nash network if costs of setting links are homogeneous and value of players is heterogeneous. They also prove that this does not hold if costs of forming links are heterogeneous. Interestingly, when we introduce homogeneous link success probabilities with identical values and costs (Bala and Goyal, 2000, [2]), the existence of equilibrium remains an open question. However, with the introduction of heterogeneity of any type in imperfect reliability models, it is possible to show that a Nash equilibrium does not exist for all parameter values.

In a model with decay with homogeneous parameters we find that Nash equilibrium always exists. However, if we introduce costs heterogeneity, with either value or decay heterogeneity, then Nash networks do not always exist. Once again we see that in models involving alternative paths, non-existence is likely. Further, it is also clear that models that incorporate heterogeneity in costs of link formation have to be more cautious about existence issues.

Finally, it is important to ask whether the richness of results stems from degrees of freedom in choosing model parameters. Instead of reiterating the comprehensive discussion in Haller and Sarangi (2005, [6]) we just summarize the main points. Although the degrees of freedom in the parameters are important, note that many of the key results of this paper can be obtained with a small set of parameter values. Thus number of parameters, the degrees of freedom in choosing and the model set up all affect outcomes in heterogeneous Nash networks.

| | **Strict Nash networks** | **Efficient Networks** | **Existence** |
|---|---|---|---|
| **Models with perfect reliability and no decay** | | | |
| Homogeneous Values and Costs | Empty network, Center-sponsored stars | Empty network, Center-sponsored stars Empty network, Minimally connected networks | Yes |
| Heterogeneous Values only | Empty network and minimal networks in which every non-singleton component is a Center-sponsored star | Empty network, Minimal networks | Yes |
| Heterogeneous Costs only | Empty networks, Minimal networks | Empty network, Minimal networks | No |
| **Imperfect Reliability Models** | | | |
| Homogeneous Values, Costs and Reliability | Empty network, Connected networks | Empty network, Connected networks | unresolved |
| Homogeneous Values, Costs and Heterogeneous reliability | Essential networks | Essential networks | No |
| Hetrogeneous Values and/or Costs and Homogeneous reliability | Essential networks | Essential networks | No |
| **Decay Models** | | | |
| Homogeneous decay and Homogeneous Values, Costs | Empty network, Connected networks | Empty network, Star networks, and Complete network | Yes |
| Homogeneous Decay and Heterogeneous values (or costs) | Essential networks | Essential networks | No |
| Heterogeneous Decay and Homogeneous values (or costs) | Essential networks | Essential networks | unresolved |
| Heterogeneous Decay and Costs, either homogeneous or heterogeneous Values | Essential networks | Essential networks | No |

Table 1: Two-way flow models: Results.

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