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Heterogeneity and Link Imperfections in Nash Networks*

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Abstract

Heterogeneity in Nash networks with two-way flow can arise due to differences in the following four variables: (i) the value of information held by players, (ii) the rate at which information decays as it traverses the network, (iii) the probability with which a link transmits information, and (iv) the cost of forming a link. Observe that the second and third forms of heterogeneity are also instances of link imperfections. In sharp contrast to the homogeneous cases in this paper we show that for any type of link imperfection, under heterogeneity involving only two degrees of freedom, all networks can be supported as Nash or efficient. To address the question of conflict between stability and efficiency, we then identify conditions under which efficient networks are also Nash. We also find that cost heterogeneity leads to non-existence of Nash networks in models with and without link imperfections. We show that in general there is no relationship between the decay and probabilistic models of network formation. Finally, we show that on reducing heterogeneity the earlier "anything goes" result disappears.

JEL Classification: C72, D85.

Key Words: Strategic reliability, decay, two-way flow models

1 Introduction

"The predicted equilibrium networks are often quite stark in nature (stars, complete networks, interlinked stars, etc.). This is partly due to the fact that most of the models that have been solved have strong symmetries in the assumed payoff functions. Without any natural heterogeneity in the problem, it is not surprising that very simple network structures emerge as predictions." – Jackson (2006, [8]). This remarkably accurate observation about strategic models of network formation implies that such models have limited applicability. In this paper we introduce heterogeneity and link imperfections in game-theoretic models of network formation to address the above mentioned problem while also creating a richer set of models.

Incorporating heterogeneity in network models leads to formulations that are realistic, and therefore are also able to explain reality better. Since results about strict Nash (and efficient) networks are directly linked to the value of parameters, heterogeneity yields a richer set of results. Moreover, it acts as a robustness check for the Nash equilibrium and efficiency results obtained using homogeneous parameters besides having significant implications for the existence of Nash equilibria. As for link imperfections, they provide an additional degree of realism to models of Nash networks. They can be used to capture the idea that links may fail to transmit information (probabilistic models of network formation) or may transmit only a part of the information (decay models of network formation).[1]

The notion of Nash networks was introduced by Bala and Goyal in two papers (2000, [1], [2]).[2] In their first paper (2000, [1]), links in the network never fail, and always transmit all information reliably. Given that link formation is costly, the authors find that Nash networks are always minimally connected. The paper also introduces link imperfection in the form of information decay whereby direct links convey more information than indirect links. In their second paper (2000, [2]), another form of link imperfection is introduced. Each link is allowed to fail with some probability p. Although link formation is still a costly act, in this

[1] When making a telephone call for instance, one may not always be able to reach the other person or when the other person can be successfully contacted, they may choose not to reveal all the information.

[2] In the following, we use Nash networks to refer to networks that satisfy Nash equilibrium as the stability concept instead of Jackson and Wolinsky's (1996, [9]) notion of pairwise stability.

case as opposed to perfectly reliable links they find that as the amount of information at stake increases, players attempt to insure themselves against link failure by forming super-connected networks, i.e, networks where links have back-ups. A key feature of both papers is that all utilized parameters are homogeneous.

In a recent paper in *Games and Economic Behavior*, Galeotti, Goyal and Kamphorst (2006, [5]) examine heterogeneity in Nash networks without taking any link imperfections into account. Their results are similar in spirit to those of Bala and Goyal (2000, [1]) in the sense that equilibrium networks now have components that are minimally connected. Heterogeneity in the presence of link imperfections has been analyzed by Haller and Sarangi (2005, [7]) who find that the homogeneity of the parameters plays a significant role in the two widely divergent results of Bala and Goyal (2000, [1], [2]). Haller and Sarangi (2005, [7]) allow different links to have different success probabilities and find that for any network $\boldsymbol{g}$, there exists a set of parameter values under which $\boldsymbol{g}$ is Nash – the model with heterogeneity can encompass the results of both Bala and Goyal papers.

In this paper we examine different possible heterogeneous Nash network formulations using the popular "connections model" introduced by Jackson and Wolinsky (1996, [9]) and studied extensively by Bala and Goyal (2000, [1], [2]). In the typical model players are endowed with some information which can be accessed by other players forming links with them. Link formation is costly, and the cost of establishing a link is incurred by the initiating player. In this model heterogeneity manifests itself in the payoff function and can occur through four different variables: (i) the value of information held by players, (ii) the rate at which information decays or loses value as it traverses the network, (iii) the probability with which a link transmits information, and (iv) the cost of forming a link. Thus by introducing heterogeneity and link imperfections in this manner we are able to generalize the results of Bala and Goyal (2000, [1], [2]) where heterogeneity is not taken into consideration, and Galeotti, Goyal and Kamphorst (2006, [5]) where link imperfections are not present.

We focus on the two-way flow models introduced by Bala and Goyal (2000, [1]).[3] The

[3]For the impact of heterogeneity on Nash and efficient networks in one-way flow models, see Galeotti (2006, [4]). For existence issues in such models we refer the reader to Billand, Bravard and Sarangi (2007, [3]).

two-way flow model allows bi-directional flow of information through a link regardless of who establishes it. Here we examine Nash networks, efficient networks and the existence of strict Nash networks under different possible heterogeneous frameworks arising from combinations of the above variables. Our main results can be summarized as follows:

- cost heterogeneity affects the existence of Nash networks with or without link imperfections, and
- under heterogeneity and link imperfections it is possible to support any network as a strict Nash or efficient network.

Cost heterogeneity leads to non-existence because it provides opportunities for link substitution. This leads to cyclical behavior since agents keep switching links. Moreover, in order to support any network as a strict Nash network or efficient network, we find that only two degrees of freedom are needed. Thus having two sets of values is sufficient for the "anything goes" result. We also identify conditions under which efficient networks are simultaneously Nash. Finally, we show that there is no obvious relationship between the probabilistic models of network formation and those allowing for decay.

The paper is organized as follows. In Section 2, we present the model setup. Section 3 contains results about models with imperfect reliability and heterogeneity. In Section 4, we study models that incorporate heterogeneity and decay. In Section 5 we discuss the relationship between these two types of model. Section 6 provides a discussion of our results and a summary of the existing work on two-way flow models.

2 Model Setup

In this section we define the formal elements of the strategic form network formation game. Let $N = \{1, \ldots, n\}$, $n \geq 3$, denote the set of with generic elements i, j, k. For ordered pairs $(i, j) \in N \times N$, the shorthand notation $i\,j$ is used and for non-ordered pairs $\{i, j\} \subset N$ the shorthand $[i\,j]$ is used.

Strategies. For player i a pure strategy is a vector $\boldsymbol{g}_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n}) \in \{0,1\}^{N \setminus \{i\}}$. Since our aim is to model network formation, $g_{i,j} = 1$ implies that there exists a direct link between i and j initiated by player i, whereas $g_{i,j} = 0$ means that i does not initiate this link. Regardless of what player i does, player j can always choose to initiate a link with i or set $g_{j,i} = 0$. We focus only on pure strategies. The set of all pure strategies of player i is denoted by $\mathcal{G}_i$ and consists of 2^{n-1} elements. The joint strategy space is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. Note that there is a one-to-one correspondence between $\mathcal{G}$ and the set of all directed graphs or networks with vertex set N. Namely, to a strategy profile $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n) \in \mathcal{G}$ corresponds the graph $(N, E(\boldsymbol{g}))$ with edge set $E(\boldsymbol{g}) = \{(i,j) \in N \times N \mid i \neq j,\ g_{i,j} = 1\}$. In the sequel, we identify a joint strategy $\boldsymbol{g}$ by its corresponding graph and use the terminology directed graph or directed network $\boldsymbol{g}$ for it.

Payoffs. Payoffs of player i are given by the difference between benefits $B_i(\boldsymbol{g})$ and costs $c_i(\boldsymbol{g})$. Hence the payoff of player i in network $\boldsymbol{g}$ is given by

$$u_i(\boldsymbol{g}) = B_i(\boldsymbol{g}) - c_i(\boldsymbol{g}). \tag{1}$$

Next we define various types of heterogeneity in networks by introducing different cost and benefit formulations.

2.1 Link Costs

Players incur costs only for the direct links they establish. We consider two possible kinds of costs.

1. Homogeneous costs. Each player i incurs a cost $c > 0$ when she initiates the direct link $i\,j$, i.e., if $g_{i,j} = 1$. Hence the total costs incurred by player i when network $\boldsymbol{g}$ is formed are given by:
$$c_i(\boldsymbol{g}) = c \cdot \sum_{j \neq i} g_{i,j}. \tag{2}$$
2. Heterogeneous costs. The cost of each link now depends on the specific pair of players who form the link. We can write the cost function of player i as:
$$c_i(\boldsymbol{g}) = \sum_{j \neq i} g_{i,j} \cdot c_{i,j}. \tag{3}$$

2.2 Link Benefits

In any given network, benefits depend on values possessed by players, the reliability of links and the nature of information decay in the model. Note that reliability and decay models are treated as mutually exclusive scenarios in the networks literature and here we retain this distinction. In a model with perfect reliability and no decay, a player obtains the full value of information from all the players she "observes" both through her direct and indirect links.[4] In a model with imperfect reliability, information obtained from a player through direct or indirect links has the same value, but links may fail to transmit information with a certain probability. To compute the value of information i acquires from j, it is now necessary to take into account all possible pathways that link i with j. This captures the idea that when one channel of communication fails to deliver the information it may be obtainable through another path. In decay models however, links always transmit the information, but information acquired through indirect links is less valuable. Since information loses value as it travels along a sequence of links it captures the idea that "it is better to have the facts straight from the horse's mouth". In this scenario, instead of considering all possible paths between two players we only consider the path that involves the least possible information decay.

2.2.1 Benefits with Perfectly Reliable Links and No Decay

In such a model, a link between players i and j allows for **two-way flow of information**. So the benefits from network $\boldsymbol{g}$ are derived from its closure $\overline{\boldsymbol{g}} \in \mathcal{G}$, defined by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for $i \neq j$. Moreover, since information is acquired through direct and indirect links we say information flows from player j to player i, when i and j are linked by means of a path in $\overline{\boldsymbol{g}}$. A **path** of length m in $\boldsymbol{g} \in \mathcal{G}$ from player i to player $j \neq i$, is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct players such that $i_0 = i$, $i_m = j$, and $g_{i_k, i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. Let $\mathcal{C}_{i,j}(\boldsymbol{g})$ be the set of paths from j to i in the network $\boldsymbol{g}$, and let $C_{i,j}(\boldsymbol{g})$ be a typical element of

[4] We say a player observes another player *directly if the geodesic distance between them is one, and observes her indirectly if the geodesic distance betwen them exceeds one.*

$\mathcal{C}_{i,j}(\boldsymbol{g})$. We denote by

$$N_i(\boldsymbol{g}) = \{j \in N |\ j \neq i, \text{ there exists a path in } \overline{\boldsymbol{g}} \text{ between } i \text{ and } j\},$$

the set of other players whom i can access or "observe" in network $\boldsymbol{g}$. Information received from j is worth $V_{i,j}$ to player i. Therefore, player i's benefits from a network $\boldsymbol{g}$ is given by:

$$B_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{i,j}. \tag{4}$$

Substituting this in (1) and using different cost formulations we obtain a class of models developed by Galeotti, Goyal, and Kamphorst (2006, [5]). Assuming homogeneous values and costs reduces this to one of the models examined in Bala and Goyal (2000, [1]).

Note that $\overline{\boldsymbol{g}}$ belongs to the set $\mathcal{H} = \{\boldsymbol{h} \in \mathcal{G} | h_{i,j} = h_{j,i} \text{ for } i \neq j\}$. There is a one-to-one correspondence between the elements of $\mathcal{H}$ and the non-directed networks with node set N. Namely, for $\boldsymbol{h} \in \mathcal{H}$ and $i \neq j$, $[i\ j]$ is an edge of the corresponding non-directed network if and only if $h_{i,j} = h_{j,i} = 1$. In what follows, we identify $\boldsymbol{h}$ with the corresponding non-directed network. Hence, the notation $[i\ j] \in \boldsymbol{h}$ stands for "$[i\ j]$ is an edge of $\boldsymbol{h}$". Also, for $\boldsymbol{k} \in \mathcal{H}$, $\boldsymbol{k} \subset \boldsymbol{h}$ means that $\boldsymbol{k}$ is a subnetwork of $\boldsymbol{h}$.

2.2.2 Benefits with Imperfectly Reliable Links

In such a model heterogeneity in benefits from links depends on the value parameter $V_{i,j} > 0$, and the probability of link success, $p_{i,j} > 0$. The closure of the network $\overline{\boldsymbol{g}}$ determines the possible flow of information in this setting. If $\overline{g}_{i,j} = 0$, then there is no (direct) information flow between i and j. If $\overline{g}_{i,j} = 1$, then the link succeeds (there is direct two-way information flow between i and j) with probability $p_{i,j} \in (0, 1)$ and fails (there is no direct information flow between i and j) with probability $1 - p_{i,j}$. Of course it is also possible that all links have the same success probability $p > 0$. Further it is assumed that success probabilities across links are independent events.

Observe that the joint strategy $\boldsymbol{g}$ gives rise to a random network with possibly different realization probabilities for different edges. These possible realizations of the random network consist of the non-directed networks $\boldsymbol{h}$ satisfying $\boldsymbol{h} \subset \overline{\boldsymbol{g}}$. Invoking the independence assumption, given $\boldsymbol{g}$, the probability of the network $\boldsymbol{h} \subset \overline{\boldsymbol{g}}$ being realized is:

$$\lambda(\boldsymbol{h} \mid \boldsymbol{g}) = \prod_{[i\ j] \in \boldsymbol{h}} p_{i,j} \prod_{[i\ j] \in \overline{\boldsymbol{g}} \setminus \boldsymbol{h}} (1 - p_{i,j}). \tag{5}$$

Note that this conditional probability can be defined similarly when all links have the same probability of success. Given a strategy profile $\boldsymbol{g}$, player i's expected benefits from the resulting random network are:

$$B_i(\boldsymbol{g}) = \sum_{\boldsymbol{h} \subset \boldsymbol{g}} \lambda(\boldsymbol{h} \mid \boldsymbol{g}) b_i(\boldsymbol{h}). \tag{6}$$

A realization of the network $\boldsymbol{h}$, which occurs with probability $\lambda(\boldsymbol{h} \mid \boldsymbol{g})$, gives player i benefits $b_i(\boldsymbol{h})$. Summing over all possible realizations $\boldsymbol{h} \subset \boldsymbol{g}$ yields expected benefits. Variations of the expected benefits formulation can be substituted in the payoff function to obtain different models.

The model of imperfect reliability where all values, costs and probabilities are identical across players and links was analyzed by Bala and Goyal (2000, [2]). We call this the **model of imperfect reliability with homogeneous parameters.** The payoff function where the link failure probability is different for each link but V and c are identical across all players is called the **model of imperfect reliability with heterogeneous links**. Such a model was first analyzed by Haller and Sarangi (2005). In this paper, we introduce a model with identical link failure probabilities, but with values $V_{i,j}$. We call this the **model of imperfect reliability with heterogeneous players**. Our paper also permits heterogeneity in link formation costs.

2.2.3 Benefits with Decay of Value through Links

Decay models were introduced by Jackson and Wolinsky (1996, [9]) under the name of the "connections model". In the Nash networks setting they were analyzed by Bala and Goyal (2000, [1]) who assumed that the value of information, the costs of link formation, and the decay parameter were identical across all players and links. In other words, they analyzed the case of homogeneous decay. We propose two different frameworks to capture heterogeneity in models with decay.

Decay with Heterogeneous Players. Here we utilize the homogeneous decay assumption in conjunction with the heterogeneous players framework of Galeotti, Goyal and Kamphorst (2006, [5]), i.e., we assume that there exists $(i, j) \neq (k, \ell)$ such that $V_{i,j} \neq V_{k,\ell}$. Then the payoff function can be written as:

$$u_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \delta^{d_{i,j}(\boldsymbol{g})} V_{i,j} - \sum_{j \in N \setminus \{i\}} c_{i,j} \cdot g_{i,j}, \tag{7}$$

where δ is the decay parameter and $d_{i,j}(\boldsymbol{g})$ is the distance in the shortest path between i and j in $\boldsymbol{g}$.[5] We use this label for the model regardless of whether link costs are homogeneous or heterogeneous.

Decay with Heterogeneous Links. Here we assume that decay associated with the link $[i\ j]$ is not identical to decay associated with the link $[\ell\ k]$ for $[\ell\ k] \neq [i\ j]$. This assumption captures the fact that under decay the quantity of information a link can convey is not the same across all links. In other words, some links or paths are "better" than others.

We measure decay associated with a link $[i\ j]$ by the parameter $\delta_{i,j} \in (0, 1)$. Given a network $\boldsymbol{g}$, it is assumed that if player i has a link with player j, then she receives information of value $\delta_{i,j}$ from j. For this model we retain the symmetry assumption, that is $\delta_{i,j} = \delta_{j,i}$. Without loss of generality we assume that the value of a link is $V = 1$. The payoff of player i in the network $\boldsymbol{g}$ is then given by:

$$u_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \left(\prod_{[\ell\ k] \in C^*_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right) - \sum_{j \in N \setminus \{i\}} c_{i,j} \cdot g_{i,j}, \tag{8}$$

where $C^*_{i,j}(\boldsymbol{g}) = \arg\max_{C_{i,j}(\boldsymbol{g}) \in \mathcal{C}_{i,j}(\boldsymbol{g})} \left\{ \prod_{[\ell\ k] \in C_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right\}$.

This expression fundamentally differs from the previous one because it does not use the geodesic distance between players to determine the value of information obtained. We use this label for the model regardless of whether link costs are homogeneous or heterogeneous.

2.3 Network Definitions

Nash Networks. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network that remains when all of player i's links have been removed. Clearly, $\boldsymbol{g} = \boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}$, where the symbol $\oplus$ indicates that

[5] In Bala and Goyal (2000, [1]), it is assumed that players can always access their own information.

$\boldsymbol{g}$ is composed of the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$ (similarly the symbol $\ominus$ is used to indicate removal of a link). A strategy $\boldsymbol{g}_i$ is a **best response** of player i to $\boldsymbol{g}_{-i}$ if

$$u_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq u_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{ for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$

Let $\mathcal{BR}_i(\boldsymbol{g}_{-i})$ denote the set of player i's best responses to $\boldsymbol{g}_{-i}$. A network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n)$ is said to be a **Nash network** if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each $i \in N$. A strict Nash network is a network where all players are playing a strict best response.

Efficient Networks. A network $\boldsymbol{g}$ is efficient if the total utility of players is maximum, that is $W(\boldsymbol{g}) = \sum_{i=1}^{n} u_i(\boldsymbol{g}) \geq \sum_{i=1}^{n} u_i(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{G}$.

Graph-theoretic Concepts. A network $\boldsymbol{g}$ is called a **star** if there is a vertex i_s, such that for all $j \neq i_s$, $\max\{g_{i_s,j}, g_{j,i_s}\} = 1$ and for all $k \notin \{i_s, j\}$, $\max\{g_{k,j}, g_{j,k}\} = 0$. Moreover a star, where $g_{i_s,j} = 1$ for all $j \neq i_s$ is a center-sponsored star, and a star, where $g_{i_s,j} = 0$ for all $j \neq i_s$, is a periphery-sponsored star. Finally, a star which is neither a center-sponsored star nor a periphery-sponsored star is a mixed star. A network $\boldsymbol{g}$ is **connected** if there is a path in $\overline{\boldsymbol{g}}$ between all players $i, j \in N$. A network $\boldsymbol{g}$ is **minimally connected** if it is connected and for all $i\,j \in E(\boldsymbol{g})$, $\boldsymbol{g} \ominus i, j$ is unconnected. A network $\boldsymbol{g}$ is **superconnected** if there exists a link $i\,j \in E(\boldsymbol{g})$ such that $\boldsymbol{g} \ominus i, j$ is still a connected network.

Finally, a network $\boldsymbol{g} \in \mathcal{G}$ is **essential** if $g_{i,j} = 1$ implies $g_{j,i} = 0$. Note that if $\boldsymbol{g} \in \mathcal{G}$ is a Nash network, then it must be essential. This follows from the fact that each link is costly while information flow is two-way and independent of which player invests in forming the link.

3 Models with Imperfect Reliability

Imperfect reliability is a link imperfection where links fail to transmit information with a positive probability. Based on the parameters three formulations of network reliability are possible. First, is the model of Bala and Goyal (2000, [2]) which is homogeneous with respect to all parameters (the pV model). This was followed by the work of Haller and Sarangi (2005, [7]) who allow for heterogeneity in link failure probabilities (the $p_{i,j}V$ model). A third kind of heterogeneity involves players with different values of information. We call this model of imperfect reliability as the model with heterogeneous players (the $pV_{i,j}$ model) and analyze it next.

We begin by illustrating that Nash equilibrium under the heterogeneous players model need not coincide with Nash equilibrium under the heterogeneous links model. To allow for meaningful comparison we impose the restriction that the expected value of a direct link is the same under both formulations, i.e., $pV_{i,j} = p_{i,j}V$, for all $(i,j) \in N \times N$.

Example 1 Let $N = \{1,2,3\}$, $V = 1$, $p = 1/5$, $c = 1/5$, and

$$\begin{pmatrix} V_{1,1} & V_{1,2} & V_{1,3} \\ V_{2,1} & V_{2,2} & V_{2,3} \\ V_{3,1} & V_{3,2} & V_{3,3} \end{pmatrix} = \begin{pmatrix} 0 & 15/4 & 9/2 \\ 15/4 & 0 & 5/3 \\ 9/2 & 5/3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ p_{2,1} & 0 & p_{2,3} \\ p_{3,1} & p_{3,2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3/4 & 9/10 \\ 3/4 & 0 & 1/3 \\ 9/10 & 1/3 & 0 \end{pmatrix}.$$

It is easy to check that $pV_{i,j} = p_{i,j}V$, for all $i,j \in N$. Straightforward calculations show that the network $\boldsymbol{g}$ with $E(\boldsymbol{g}) = \{1\,3, 3\,2, 2\,1\}$ is a Nash network in the $pV_{i,j}$ framework. But, in the $p_{i,j}V$ framework the difference between the expected utility of player 3 in $\boldsymbol{g}$ and $\boldsymbol{g}'$ with $E(\boldsymbol{g}') = \{1\,3, 2\,1\}$ is $-1/15 < 0$. Hence $\boldsymbol{g}$ is not a Nash network. Furthermore, it is easy to check that $\boldsymbol{g}'$ is a Nash network for the $p_{i,j}V$ game.

The intuition behind this diverging result is the following. Since we have $p_{i,j}V = pV_{i,j}$, the higher $p_{i,j}$ is, the higher is $V_{i,j}$ (for given V and p respectively). In the $pV_{i,j}$ framework, higher values provide an incentive to form links, while in $p_{i,j}V$ framework, higher probabilities reduce the marginal benefits of alternative paths. Thus player 3 is better off not forming the link 32.

We present two results for this model. First, instead of explicitly characterizing all the Nash equilibria, for the sake of brevity, we demonstrate that there exist conditions such that every essential network can be strict Nash. This is also shown to be true for efficient networks.

Theorem 1 *: (Imperfect reliability with heterogeneous players.) Let $\boldsymbol{g}$ be an essential network. Then, there exists a homogeneous link cost $c > 0$, a probability p, and an array $\boldsymbol{V} = [V_{i,j}]$ of values such that*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*
2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game.*

Proof Let $\boldsymbol{g}$ be an essential network.

1. First, we prove that there is a link cost $c > 0$, a probability p, and an array $\boldsymbol{V}$ of values such that $\boldsymbol{g}$ is a strict Nash network. We assume that $V_{i,j} \in \{\varepsilon, V\}$. More precisely, if $g_{i,j} = 1$, then $V_{i,j} = V$, and if $g_{i,j} = 0$, then $V_{i,j} = \varepsilon < V$.

   Let $\boldsymbol{g} \oplus i, j$ with $V_{i,j} = \varepsilon$, be the network where i obtains the largest amount of resources thanks to the link $i\, j$. Then, the amount of resources that player i obtains through j is given by:
$$\varepsilon p + \sum_{\ell=2}^{n(n-1)/2} p^{\ell}(a_{\ell}\varepsilon + a'_{\ell}V) \tag{9}$$
   where $a_{\ell}, a'_{\ell} \in \mathbb{Z}$ for each $\ell \in \{2, \ldots, n(n-1)/2\}$. It is possible that some a_{ℓ} or some a'_{ℓ} are null. Let $A = \sum_{\ell=2}^{n(n-1)/2}(|a_{\ell}| + |a'_{\ell}|)$.

   Let $\boldsymbol{g} \ominus i, j$ be the network where i obtains the resources of j with the largest probability, with $V_{i,j} = V$, given that there is no link between i and j in this network. Then, the amount of resources of j that player i obtains, in $\boldsymbol{g} \ominus i, j$, is given by:
$$\sum_{\ell=2}^{(n-1)(n-2)/2} p^{\ell} b_{\ell} V$$
   where $b_{\ell} \in \mathbb{Z}$ for each $\ell \in \{2, \ldots, (n-1)(n-2)/2\}$. It is possible that some b_{ℓ} are null. Let $B = \sum_{\ell=2}^{(n-1)(n-2)/2} |b_{\ell}|$.

   Given A and B, there exists p such that
$$p < \frac{V - \varepsilon}{V(A+B)},\ A \neq 0 \text{ or } B \neq 0,$$
   and
$$p\varepsilon + p^2 AV < c < Vp - Bp^2 V.$$

Now, it is clear that a player i who did not form a link with j in $\boldsymbol{g}$ has no incentive to form this link. Indeed, if i forms a link with j, then her payoff function increases by an amount which is bounded above by $p\varepsilon + p^2AV - c < 0$. Likewise, a player i has an incentive to preserve her link with player j if $g_{i,j} = 1$. Indeed, a player j brings to i an amount which is bounded below by $Vp - Bp^2V - c > 0$. It follows that all players play a best response in $\boldsymbol{g}$. Hence $\boldsymbol{g}$ is a strict Nash network.

2. Second, we prove that there is a link cost $c > 0$, a probability p, and an array $\boldsymbol{V}$ of values such that $\boldsymbol{g}$ is an efficient network. Again for $g_{i,j} = 1$, $V_{i,j} = V$, and if $g_{i,j} = 0$, then $V_{i,j} = \varepsilon < V/2n$. Let $\mathcal{F} = \{i\,j|g_{i,j} = g_{j,i} = 0\}$ be the set of links such that $V_{i,j} = \varepsilon$ and $\mathcal{D} \subset \mathcal{F}$. Let $\boldsymbol{g}'(\mathcal{D}) = \arg\max_{\boldsymbol{g}' \in \mathcal{G}}\{W(\boldsymbol{g}' \oplus \mathcal{D}) - W(\boldsymbol{g}')\}$ be the network where the n players obtain the largest amount of resources thanks to the links in $\mathcal{D}$. Let $\mathcal{D}^* = \arg\max_{\mathcal{D} \subset \mathcal{F}}\{\boldsymbol{g}'(\mathcal{D})\}$ be the subset of $\mathcal{F}$ which brings the maximal increasing total utility. This is at most:

$$2|\mathcal{D}^*|\varepsilon p + \sum_{\ell=2}^{n(n-1)/2} p^{\ell}(e_\ell \epsilon + e'_\ell V) \tag{10}$$

where $e_\ell, e'_\ell \in \mathbb{Z}$ for each $\ell \in \{1, \ldots, n(n-1)/2\}$. It is possible that some e_ℓ or some e'_ℓ are null. Let $E = \sum_{\ell=2}^{n(n-1)/2}(|e_\ell| + |e'_\ell|)$.

We can find p such that

$$p < \frac{V - 2n\varepsilon}{(E+B)V}, \; E \neq 0 \text{ or } B \neq 0,$$

and

$$2np\varepsilon + p^2EV < c < Vp - Bp^2V.$$

Then, by using arguments similar to those in the previous part, we conclude that $\boldsymbol{g}$ is an efficient network.

□

Corollary 1 *Let $\boldsymbol{g}$ be an essential network. Then, there exists a homogeneous link cost $c > 0$, a probability p, and an array $\boldsymbol{V} = [V_{i,j}]$ of values such that $\boldsymbol{g}$ is a strict Nash network and an efficient network.*

Proof It is sufficient to inspect the bounds for efficient and strict Nash networks given in the proof of Theorem 1 to establish the corollary. □

A few remarks are in order here. First, although Haller and Sarangi (2005, [7]) do not investigate this issue, it can be easily shown that a similar efficiency result holds for the heterogeneous links model.

Second, for any network to be Nash, it is enough to introduce two degrees of value heterogeneity. Indeed, in the proof of Theorem 1 we only need two different values. Further, even if the difference between these two values is small we can always find parameters p and c such that the network is Nash.

Third, Theorem 1 still holds if there is heterogeneity in link costs as well. A simple continuity argument can be used to verify this.

Next it is also worth asking what is the relationship between efficiency and stability in this model. Both Bala and Goyal (2000, [2]), and Haller and Sarangi (2005, [7]) claim that when costs are very high or very low, there is no conflict between stability and efficiency. We find that the same is true for the heterogeneous players model. It is possible to use a continuity argument to preserve the result of Bala and Goyal (2000, [2], pp. 223-224) concerning the conflict between Nash networks and efficient networks.

Finally, one can ask the question: Given Theorem 1, is there any role for explicit characterization of the strict Nash networks? Although for the sake of brevity we do not pursue this, it can still be a meaningful exercise since the framework allows for the existence of multiple equilibria. Consider for instance Example 1 in Bala and Goyal (2000, [2]). This is the simplest possible imperfect reliability scenario since it only has 3 players with homogeneous values, costs and reliability. First, there exists a parameter range in which periphery-sponsored stars are Nash. Clearly, as there is no rule for selecting the central player, the coordination problem for selecting this player may still persist under heterogeneity. Moreover, the authors identify a parameter range where two kinds of architectures are Nash: the center-sponsored star and mixed stars. Such an outcome is possible even with parameter heterogeneity.

We now identify sufficient conditions for the simultaneous existence of two types of star networks in equilibrium in the heterogeneous players framework. Let $V^{\overline{m}} = \max_{(i,j)\in N\times N}\{V_{i,j}\}$ and $V^{\underline{m}} = \min_{(i,j)\in N\times N}\{V_{i,j}\}$.

Proposition 1 *: Consider the heterogeneous players model with homogeneous costs. If $pV^{\underline{m}} > c$ and $(n-2)(1-p^2)V^{\overline{m}} + (1-p)V^{\overline{m}} < c$, then center-sponsored stars and mixed stars are strict Nash networks.*

Proof The proof by contradiction is an adaptation of the proof of Bala and Goyal (2000, [2]). Let $i_s \in N$ be the center of the star. We focus on center-sponsored stars. Suppose that there is a player j who is not linked with i_s ($g_{j,i_s} = g_{i_s,j} = 0$). Then the marginal payoff that i_s obtains from a player j is $pV_{i_s,j} \geq pV^{\underline{m}} > c$, a contradiction. Moreover, if player j forms $k \geq 1$ links, then her payoff is bounded above by

$$\sum_{i\in N\setminus\{j\}} V_{j,i} - kc. \tag{11}$$

The payoff that player j obtains in the center-sponsored star is:

$$pV_{j,i_s} + p^2 \sum_{i\in N\setminus\{j,i_s\}} V_{j,i}. \tag{12}$$

Subtracting (12) from (11) shows that j's maximum incremental payoff from one or more links is no larger than $(n-2)(1-p^2)V^{\overline{m}} + (1-p)V^{\overline{m}} - c$, which is negative by choice of p and c. Hence, j's best response is to form no link with a player $i \neq i_s$. The proof for the mixed stars can be made with similar arguments. □

Existence of Nash networks. We show that there exist parameters p, c, $(V_{i,j})_{(i,j)\in N\times N\setminus\{i\}}$, such that there is no Nash network in the $pV_{i,j}$ framework. Note that this result is preserved when the costs of link formation are heterogeneous.

Example 2 : *(Non-existence of Nash networks.)* Let $N = \{1,2,3\}$ be the set of players. Let $pV_{i,j} < c$ for all links $i\,j$ except the link 1 3. Also, let $pV_{21} + p^2V_{23} > \max\{c, p^2V_{2,1} + pV_{2,3}\}$, $p(1-p)(pV_{1,2}+(1+p)V_{1,3}) < c$, $p(1-p)(pV_{2,1}+(1+p)V_{2,3}) < c$, $p(1-p)(pV_{3,1}+(1+p)V_{3,2,}) > c$.

1. The empty network is not a Nash network, since $pV_{1,3} > c$.

2. A network with one link cannot be a Nash network. Indeed, if this link is the link $i\,j \neq 1\,3$, then player i can obtain a higher payoff by deleting this link, since $pV_{i,j} < c$. If this link is the link $i\,j = 1\,3$, then player 2 has a higher payoff if she forms the link 2 1, since $pV_{2,1} + p^2 V_{2,3} > c$.

3. Next, a network $\boldsymbol{g}$ with two links cannot be a Nash network. Given that Nash networks must be essential in such a network, there always exists a path in $\overline{\boldsymbol{g}}$ between the players.

   i. Since $pV_{i,j} < c$ for all links $i\,j$, except the link 1 3, no network where a link allows access to resources of only one other player can be Nash (except for the link 1 3). From this it follows that only networks with links $\{1\,3, 2\,3\}$, $\{1\,2, 3\,2\}$, $\{2\,1, 3\,1\}$ or $\{1\,3, 2\,1\}$ can be Nash.

   ii. We know that player 2 prefers the link 2 1 to the link 2 3 (since $pV_{2,1} + p^2 V_{2,3} > p^2 V_{2,1} + pV_{2,3}$). Thus, networks with links 1 3, 2 3, cannot be Nash.

   iii. We know that, every thing else being equal, player 1 prefers the link 1 3 to the link 1 2 (since $pV_{1,3} > c > pV_{1,2}$). Thus, network with links 1 2 and 3 2 cannot be Nash.

   iv. Since $p(1-p)(pV_{3,1} + (1+p)V_{3,2}) > c$, the networks with links $\{2\,1, 3\,1\}$ or $\{1\,3, 2\,1\}$ cannot be Nash. Indeed, in such a case, player 3 has an incentive to set a link with player 2.

   Hence, it follows that a network with two links cannot be a Nash network.

4. Finally, we show that a network $\boldsymbol{g}$ with three links cannot be a Nash network. Note that in this network there always exists a path in $\overline{\boldsymbol{g}}$ between the players. Since $p(1-p)(pV_{1,2} + (1+p)V_{1,3}) < c$ and $p(1-p)(pV_{2,1} + (1+p)V_{2,3}) < c$, then no network where player 1 or player 2 have formed links can be a Nash network. It follows that no network with three links can be a Nash network.

4 Models with Decay

In this section we focus on the second type of link imperfections – situations where the value of information conveyed by links decreases with network distance. Bala and Goyal (2000,

[1]) analyze such networks in a homogeneous setting. Their main emphasis was on identifying conditions that allow some architectures (complete and empty networks, and stars) to be Nash or efficient stating that it is difficult to provide a complete characterization of Nash or efficient networks. Here we analyze the consequences of heterogeneity.

4.1 Decay with Heterogeneous Agents

In this section we obtain two main results.[6] First, we demonstrate that all networks can be supported as strict Nash and efficient. Next, we show that there exist parameter values for which there is no Nash network in pure strategies.

Theorem 2 *: (Decay with heterogeneous players.) Let $\boldsymbol{g}$ be an essential network. If the payoff function satisfies equation (7), then there exist a link cost $c > 0$ and an array $\boldsymbol{V} = [V_{i,j}]$ of values such that:*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*
2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game. Moreover this network is also strict Nash.*

Proof Let $\boldsymbol{g}$ be an essential network and $V^{\overline{m}} = \max_{(i,j)\in N\times N}\{V_{i,j}\}$.

1. For $g_{i,j} = 1$, let $V_{i,j}(\delta - \delta^2) > c$, and if $g_{i,j'} = 0$, then let $c > \delta V_{i,j'} + (n-2)\delta^2 V^{\overline{m}}$, with $V_{i,j} > V_{i,j'}$. These two conditions are compatible if δ is sufficiently close to zero. Under these restrictions no player has an incentive for unilateral deviation proving the first part.
2. For $g_{i,j} = 1$, let $V_{i,j}(\delta - \delta^2) > c$, and if $g_{i,j'} = 0$, then let $c > \delta(V_{i,j'} + V_{j',i}) + \delta^2(n-2)nV^{\overline{m}}$ (we assume $g_{j',i} = 0$). Again these two conditions are compatible if δ is sufficiently close to zero and gives us the desired efficient network. Also, by inspecting the bounds chosen for efficient and strict Nash networks it is easy to see that $\boldsymbol{g}$ is also simultaneously a strict Nash network.

[6] Galeotti, Goyal and Kamphorst (2006, [5]) consider the case of small levels of decay with heterogeneous players. Their insider-outsider model only has two groups of players, and thus two possible values of δ.

□

Existence of Nash networks. In this context we begin by showing that the model of Bala and Goyal (2000, [1]), always has a Nash network. This result continues to hold if heterogeneity is not "too high", more precisely if $V_{i,j} = V_i$ for all $i \in N$.

Theorem 3 *: Suppose costs of forming links are homogeneous.*

1. *If the decay parameter and the values are homogeneous, then there always exists a Nash network.*
2. *If the payoff function satisfies equation (7) and, for all $i \in N$, $V_{i,j} = V_i$, for all $j \in N \backslash \{i\}$, then a Nash network always exists.*

Proof We now prove both parts of the theorem.

1. If $\delta V \leq c$, then the empty network is a Nash network. If $\delta V > c$, then there are two possibilities: (i) if $(\delta - \delta^2)V \leq c$, then star networks are Nash, and (ii) if $(\delta - \delta^2)V > c$, then the complete network is Nash.
2. If $\delta V_i \leq c$ for all $i \in N$, then the empty network is a Nash network. If there exist players $i \in N$ such that $\delta V_i > c$, then starting with the empty network let one of these players, say i_0, form link with all other players giving us a center-sponsored star. If for all $i \in N \setminus \{i_0\}$, $(\delta - \delta^2)V_i \leq c$, then the previous network is Nash. Otherwise, we let all players $i \in N \setminus \{i_0\}$ such that $(\delta - \delta^2)V_i > c$ be connected to all other players (obviously, in that case we choose i_0 such that $(\delta - \delta^2)V_{i_0} > c$), and we obtain a Nash network.

□

Notice that the above proof also identifies conditions under which we can obtain some simple architectures like stars as Nash networks. The following example shows that non-existence can occur when we introduce higher player heterogeneity. Moreover, this example can be modified to incorporate heterogeneity in link formation costs and still achieve the same outcome.

Example 3 *(Non-existence of Nash networks.)* Let $N = \{1, \ldots, 5\}$ be the set of players, and assume that:

1. $V_{1,2}\left(\delta-\delta^4\right)+V_{1,3}(\delta^2-\delta^3)>c$, $\delta V_{1,3}<\delta V_{1,2}<c$, and for all $j\neq 2$, $\delta V_{1,j}+\delta^2\sum_{k\neq j}V_{1,k}<c$.

2. $V_{2,3}\left(\delta-\delta^4\right)+V_{2,4}(\delta^2-\delta^3)<c$, $\delta V_{2,3}+\delta^2V_{2,4}+\delta^3V_{2,5}+\delta^4V_{2,1}>c$, and for all $j\neq 3$, $\delta V_{2,j}+\delta^2\sum_{k\neq j}V_{2,k}<c$.

3. $\delta V_{3,4}>c$ and $\delta\sum_{k\neq 4}V_{3,k}+\delta^2V_{3,4}<c$.

4. $\delta V_{4,5}>c$ and $\delta\sum_{k\neq 5}V_{4,k}+\delta^2V_{4,5}<c$.

5. $\delta V_{5,1}>c$ and $\delta\sum_{k\neq 1}V_{5,k}+\delta^2V_{5,1}<c$.

These five points provide a list of the players with whom the others have no incentives to form links, as well as those with whom they would like to form links. For example, item 1 implies that player 1 will never form a link with players 3, 4 and 5. Moreover, a Nash network must contain the links 3 4, 4 5, 5 1. From all of this, it follows that there is four possible Nash networks: $E(\boldsymbol{g}^1)=(3\,4,4\,5,5\,1,1\,2,2\,3)$, $E(\boldsymbol{g}^2)=(3\,4,4\,5,5\,1,1\,2)$, $E(\boldsymbol{g}^3)=(3\,4,4\,5,5\,1)$, $E(\boldsymbol{g}^4)=(3\,4,4\,5,5\,1,2\,3)$. We know from item 2 that player 2 prefers the network $\boldsymbol{g}^2$ to the network $\boldsymbol{g}^1$, so $\boldsymbol{g}^1$ is not Nash. Likewise, player 1 prefers the network $\boldsymbol{g}^3$ to the network $\boldsymbol{g}^2$ by point 1, so $\boldsymbol{g}^2$ is not Nash. Player 2 prefers the network $\boldsymbol{g}^4$ to the network $\boldsymbol{g}^3$ by point 2, so $\boldsymbol{g}^3$ is not Nash. Finally, by point 1, player 1 prefers the network $\boldsymbol{g}^1$ to the network $\boldsymbol{g}^4$. Hence $\boldsymbol{g}^4$ is not Nash.

4.2 Decay with Heterogeneous Links

In this section we consider situations where players have homogeneous values while the decay through each link is different. We obtain the following result.

Theorem 4 *: (Decay with heterogeneous links.) Let $\boldsymbol{g}$ be an essential network. If the payoff function satisfies equation (8) and costs of forming links are homogeneous, then there exist $c>0$ and an array $\boldsymbol{\delta}=[\delta_{i,j}]$ of decay such that:*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*

2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game. Moreover this network is also strict Nash.*

Proof

1. Let $\boldsymbol{g}$ be an essential network. For $g_{i,j} = 1$, let $c < \left(\delta_{i,j} - (\delta^m)^2\right) V$, where $\delta^m = \max_{(i',j') \in N^2} \{\delta_{i',j'}\}$. Also, for $g_{i,j} = 0$, let $c > (\delta_{i,j} + \delta^m \delta_{i,j}(n-2))V$. It can be verified that these two conditions are compatible. Under these conditions a player i, who has a link with a player j, has no incentive to remove it. Indeed, player i has the greatest incentive to delete a link with j when she obtains the resources of j from a player k such that $\max\{g_{k,i}, g_{i,k}\} = \max\{g_{k,j}, g_{j,k}\} = 1$. The condition which allows player i to maintain her link with j is: $c < (\delta_{i,j} - \delta_{i,k}\delta_{j,k}) V$ which is always true if $c < \left(\delta_{i,j} - (\delta^m)^2\right) V$. Likewise, a player i who does not have a link with a player j has no incentive to form this link. Indeed, a player j can provide at most information of value $(\delta_{i,j} + \delta^m \delta_{i,j}(n-2))V$ to player i.

2. The proof of the second part of the proposition is similar to the previous part, but now we assume that if $g_{i,j} = 0$, then $c > \delta_{i,j} + \delta_{j,i} + (\delta^m)^2 (n-2)nV$ (given that $g_{j,i} = 0$). Since the two conditions are again compatible, we can conclude that $\boldsymbol{g}$ is an efficient network. Finally, by inspecting the bounds for efficient and strict Nash networks it is easy to see that $\boldsymbol{g}$ is also simultaneously a strict Nash network.

□

Next, in contrast to the connections model, if we assume that decay begins only with indirect neighbors (instead of direct neighbors), then we can show that regardless of the value of the parameters, some essential networks are neither Nash nor efficient.

Example 4 Let $N = \{1, 2, 3\}$ be the set of players, let $\boldsymbol{g}$ be a network such that $E(\boldsymbol{g}) = \{1\ 2\}$. Then, $\boldsymbol{g}$ is not a Nash network. Indeed, if player 1 has an incentive to form a link with player 2, then $V < c$. In that case, player 3 has an incentive to form a link with player 1. Likewise $\boldsymbol{g}$ is not an efficient network.

Existence of Nash equilibrium. The existence of Nash equilibria in the models with decay and heterogeneous links remains an open question. However, if we add the heterogeneity of costs to the heterogeneity of links, then it is possible to adapt an argument from Haller, Kamphorst and Sarangi (2006, [6]) to show that Nash equilibria do not always exist. Indeed, they show in example 2 (pg.602) that there exist situations with no decay ($\delta = 1$), homogeneous values and heterogeneous costs, such that there does not exist any Nash network. Hence, by continuity it is possible to construct a similar example with δ sufficiently close to 1 where Nash equilibria will not exist.

5 Comparing Probabilistic and Decay Models

In this section the two models of link imperfections are compared. Specifically, we focus on two questions: Can strict Nash networks in one class of models tell us anything about strict Nash networks in the other class of models? Further, what happens in probabilistic and decay models if we reduce the extent of heterogeneity?

5.1 Relationship between Probabilistic and Decay Models

The probabilistic model uses all the paths between two players for computing payoffs while the decay model only uses the shortest path between two players to determine payoffs. At first glance this suggests that decay models might be a subset of the probabilistic models. Hence we ask if information about strict Nash networks in probabilistic models give some sense about strict Nash networks in decay models. To address this question, we compare marginal payoffs of links in both types of models.[7] In order to make the models comparable we assume that starting from the empty network, the marginal payoffs of a link is the same in both models, that is we set $\delta = p$.

We first show that when the initial network is minimal, the marginal payoff of a link is always at least as great in the probabilistic model as in the decay model.

Indeed, suppose that in a minimal network $\boldsymbol{g}^1$, one player, say player i, forms a link with say player j. Let the resulting network be denoted by $\boldsymbol{g}^2$. Either j is not observed by player i in $\boldsymbol{g}^1$ and it is obvious that the marginal payoff of the link $i\,j$ is the same in both models, or j is

[7]Note that in the probabilistic model players' marginal payoffs are expected marginal payoffs. However, for both types of models, we use the term marginal payoffs to make reading easier. Moreover, we assume that players in the probabilistic model are risk neutral.

observed by i in $\boldsymbol{g}^1$. In the latter case, in the decay model, player i being at distance 1 from player j in $\boldsymbol{g}^2$, she obtains an amount p of resources of player j. In the probabilistic model, i accesses to the resources of j in $\boldsymbol{g}^2$ if the link $i\,j$ works, that occurs with a probability p. She also accesses the resources of j even when the link $i\,j$ does not work. It is enough that all the links which were contained in a path from j to i in $\boldsymbol{g}^1$ work. So, the amount of resources of player j obtained by player i in $\boldsymbol{g}^2$ is greater than p. With the same type of reasoning, we can show that the part of the resources of players $k \neq j$, obtained by i in $\boldsymbol{g}^2$, is at least as great in the probabilistic model as in the decay model. The result follows. From this result, it is straightforward that *a minimally connected Nash network in the probabilistic model is also a Nash network in the decay model.*

Next what happens if the initial network is not minimal? The example which follows shows that the above result does not hold anymore.

Example 5 Let $N = \{1, 2, 3, 4\}$ be the set of players and let $\boldsymbol{g}^1$ be a network such that $E\{\boldsymbol{g}^1\} = (1\,2, 2\,3, 3\,4, 4\,1)$.

Suppose that in $\boldsymbol{g}^1$ player 1 forms a link with player 3. We can check that for some p, for instance $p = 0.8$, the marginal payoff of this link is greater in the probabilistic model, whereas the converse is true for some other p, for instance $p = 0.9$.

Recall that if the initial network is minimal, the marginal payoff of a link is always as great in the probabilistic model as in the decay model. This difference in the result can be explained as follows.

Suppose that the initial network, denoted by $\boldsymbol{g}^1$, is not minimal. Then, there exist at least two players in $\boldsymbol{g}^1$, say i and j, such that there are at least two paths between these two players. Let player i form a link with player j in $\boldsymbol{g}^1$ and denote by $\boldsymbol{g}^2$ the resulting network. Although the total payoff of player i in $\boldsymbol{g}^2$ is greater in the probabilistic model than in the decay model, this does not imply that the marginal payoff of the link $i\,j$ is greater in the probabilistic model than in the decay model. Indeed, it is easy to check that, in $\boldsymbol{g}^1$, player i also gets a greater payoff in the probabilistic model than in the decay model.

When the initial network is not minimal, the difference in the marginal payoff of a link $i\,j$

depends on the architecture of the initial network (in particular the number of paths that exist between player i and the other players from whom i obtains resources) and on the probability that a link works. This makes it difficult to find a general rule which orders the marginal payoff of a link in both models. *Thus, when the number of players is greater than* 3 *and the initial network is not minimal, information about strict Nash networks in one type of models does not provide any indication about strict Nash networks in the other type of models.*

5.2 Consequences of Reducing Heterogeneity

Until now we have considered heterogeneity in values to be dependent on the pair of players. This leads to an "anything goes" result in both types of models with link imperfections. Now we ask what happens if heterogeneity in values is simply player dependent.[8] In particular, if $V_{i,j} = V_i$ or $V_{i,j} = V_j$ for all players $i \in N$. For instance, let $N = \{1, 2, 3\}$ be the set of players, $V_{i,j} = V_i$ for all $i \in N$, and assume a network $\boldsymbol{g}$ where $g_{1,2} = 1$ and $g_{i,j} = 0$ for all $i, j \neq 1, 2$. It is obvious that either player 1 has an incentive to delete the link $1, 2$ or has an incentive to form the link $1, 3$ under the probabilistic or decay models. Hence $\boldsymbol{g}$ can never be a Nash network.

Next consider $V_{i,j} = V_j$ for all $i \in N$, and assume a network $\boldsymbol{g}$ where $g_{1,2} = 1$ and $g_{i,j} = 0$ for all $i, j \neq 1, 2$. Now either player 1 has an incentive to delete the link $1, 2$ or player 3 an incentive to form the link $3, 2$. Again $\boldsymbol{g}$ can never be a Nash network under the probabilistic or decay models. Thus we find that on switching from link based heterogeneity to player based heterogeneity the "anything goes" result does not hold anymore.

6 Discussion

We now sum up the main insights obtained from the introduction of heterogeneity. Additionally Table 1 provides an overview of results in two-way flow models with and without heterogeneity. The first column here indicates the scope of strict Nash networks and the second column does the same for efficient networks.

[8] We are thankful to Matt Jackson for suggesting this.

First, we find that the introduction of heterogeneity allows for an "anything goes" result in models with link imperfections. Not surprisingly, it holds for both strict Nash and efficient networks. This is shown to be true for both models with heterogeneous links as well as heterogeneous players.

Second, the conditions which are presented to make any network efficient are also sufficient for them to be strict Nash. Thus we are able to identify conditions which simultaneously make efficient networks strict Nash addressing the important issue of conflict between stability and efficiency.

Third, it is important to ask whether the richness of our strict Nash networks stems from the degrees of freedom available in choosing model parameters made possible by heterogeneity.[9] First, it is easy to verify that the introduction of heterogeneity in only one of the parameters (values, costs, reliability, decay) dramatically increases the set of Nash networks. Secondly however, a quick look at the results of Galeotti, Goyal and Kamphorst (2006, [5]) is enough to show that having many degrees of freedom does not always lead to an "anything goes" result. In fact as long as we have link imperfections we only need to impose minimal requirements in term of the degrees of freedom. All our proofs only require two degrees of freedom in choosing the value of the relevant parameters in models with heterogeneous links and heterogeneous players.[10] It is also important to bear in mind that cost heterogeneity is not required to obtain this result. Thus one main insight that emerges from this paper is that the stark and simplistic networks found in the different homogeneous parameter formulations are very special cases and not robust to the introduction of different types of heterogeneity.

In Table 1 when going from a deterministic model with homogeneous parameters to a homogeneous probabilistic link failure model, Bala and Goyal (2000 [1], [2]) find that Nash networks change from being minimally connected to super-connected. More precisely, they find that the strict Nash networks change from being empty and center-sponsored stars to being empty and connected networks. This is also true when we allow for decay with homogeneous parameters. In the Galeotti, Goyal and Kamphorst (2006, [5]) formulation, that allows for heterogeneity

[9] See also Haller and Sarangi (2005, [7]) for a discussion of this.

[10] Moreover, the result holds as long as there is some difference between these two values.

in values and costs in the deterministic framework, empty and minimal networks with center-sponsored stars can be supported as Nash. Moreover, the authors show that only minimal networks can be Nash. In contrast, heterogeneity in imperfect reliability models, whether it is of the heterogeneous links type à la Haller and Sarangi (2005, [7]), or heterogeneous players type as shown in this paper, always yields an "anything goes" result implying that any network can be sustained as strict Nash by an appropriate set of parameters. We find that the same analysis holds when we introduce heterogeneity in models with decay. Moreover, Table 1 shows that a similar trend holds for efficiency. The key insight here is that when heterogeneity requires players to take into account alternative paths between players instead of just affecting values and costs of link formation, then it is possible to obtain a richer set of networks.

Fourth, we find that cost heterogeneity leads to non-existence of Nash equilibria both in models with and without link imperfections. Additionally we find that under value heterogeneity a Nash equilibrium always exists in these different models. The underlying intuition is the fact that cost heterogeneity creates link substitution possibilities which can lead to cyclical behaviors.

The last column in Table 1 summarizes the results regarding existence of strict Nash networks from different models. Bala and Goyal (2000, [1]) show the existence of strict Nash networks in the homogeneous deterministic framework through a constructive proof. Haller, Kamphorst and Sarangi (2007, [6]) show that in the deterministic setting there always exists a Nash network if costs of setting links are homogeneous and values of players are heterogeneous. They also prove that this result does not hold if costs of forming links are heterogeneous. Interestingly, with homogeneous link success probabilities under identical values and costs (Bala and Goyal, 2000, [2]), the existence of strict Nash networks remains an open question. However we believe that a strict Nash network will always exist in this instance. As can be seen in the table with the introduction of heterogeneity of any type in imperfect reliability models, it is possible to show that a strict Nash equilibrium does not exist for all parameters values.

Next in the model of decay with homogeneous parameters we find that strict Nash equilibrium always exists. In the model with heterogeneous players we are able to show non-existence by means of an example, while in the model with heterogeneous links the existence of strict Nash networks remains an open question. However, for both the unresolved existence problems in Table 1, allowing costs heterogeneity is sufficient to show non-existence. This can be done by adapting a proof from Haller, Kamphorst and Sarangi (2007, [6], pg.602). Thus, as already noted, we see that in models involving alternative paths, non-existence is likely. It follows therefore that in models allowing for heterogeneity in costs of link formation we have to be cautious about existence issues.

Fifth, we find that on comparing the two different types of models with link imperfections it is not easy to make general statements. The most we can say is that a minimally connected network that is a strict Nash network in the probabilistic model is also a strict Nash network in the decay model. Finally, we also find that making the heterogeneity player dependent is sufficient to preclude the "anything goes" result regarding efficient and strict Nash networks in both classes of models, once again highlighting the crucial role of link imperfections.

	Strict Nash networks	**Efficient Networks**	**Existence**
Models with perfect reliability and no decay			
Homogeneous Values and Costs	Empty network, Center-sponsored stars	Empty network, Minimal networks	Yes
Heterogeneous Values only	Empty network and minimal networks in which every non-singleton component is a Center-sponsored star	Empty network, Minimal networks	Yes
Heterogeneous Costs only	Empty networks, Minimal networks	Empty network, Minimal networks	No
Imperfect Reliability Models			
Homogeneous Values, Costs and Reliability	Empty network, Connected networks	Unresolved	Unresolved
Homogeneous Values, Costs and Heterogeneous reliability	Essential networks	Essential networks	No
Heterogeneous Values and/or Costs and Homogeneous reliability	Essential networks	Essential networks	No
Decay Models			
Homogeneous decay and Homogeneous Values, Costs	Empty network, Connected networks	Empty network, Star networks, and Complete network	Yes
Homogeneous Decay and Heterogeneous values (or costs)	Essential networks	Essential networks	No
Heterogeneous Decay and Homogeneous values (or costs)	Essential networks	Essential networks	Unresolved
Heterogeneous Decay and Costs, either homogeneous or heterogeneous Values	Essential networks	Essential networks	No

Table 1: Two-way flow models: Results.

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