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Abstract

A preference-theoretic characterization of weak complementarity is provided based on an explicit representation of revealed preference. Weak complementarity is defined in terms of the observable property of nonessentiality and the unobservable property of no existence value. Preference-theoretic characterizations of these properties facilitate a precision and intuition that is not generally available within the existing calculus-based literature. An exact welfare measure is specified that does not require a continuous nonmarket good or monotonic preference on the nonmarket good, and which can be easily generalized to accommodate infinite choke prices. It is shown that no existence value can be rejected by revealed preference, contradicting a widely stated assertion within the literature. Even though no existence value is unobservable, it does require an observable condition that is nontrivial with three or more market goods.

Keywords: Weak complementarity, Preference-theoretic, Testing preference restrictions, Nonessentiality, No existence value, Discrete nonmarket goods, Infinite choke prices.

1 Introduction

A core methodological problem of nonmarket goods is that revealed preference from market demand does not by itself provide sufficient information for welfare analysis. There are several diverse methodological approaches for providing the necessary additional preference information. One approach involves imposing intuitively appealing preference assumptions, typically called "preference restrictions" or "maintained hypotheses." The principal example is weak complementarity as introduced by Mäler.[1] Weak complementarity has always been implemented with calculus-based techniques. In this paper I instead use a preference-theoretic methodology to characterize weak complementarity based on an explicit representation of revealed preference. Weak complementarity is defined in terms of the observable property of nonessentiality and the unobservable property of no existence value. The preference-theoretic characterizations of these properties facilitate a precision and intuition that is not generally available with the existing calculus-based literature. An exact welfare measure is specified that does not require a continuous nonmarket good or monotonic preference on the nonmarket good, and which can also be easily generalized to accommodate infinite choke prices. When there are three or more market goods, it is shown that the unobservable assumption of no existence value can be rejected by revealed preference.

I begin by establishing the ties between a preference-theoretic approach and the usual calculus treatment with utility functions. Let z represent a nonmarket good that can take on any values of the set Z. A nonmarket good may be discrete or continuous, and the values it takes on might be scalars, vectors, or even non-numerical attributes such as $Z = \{\text{Poor Fishery}, \text{ Thriving Fishery}\}$.[2] Superscripts are used to distinguish individual elements of Z, as in $z^a, z^b \in Z$. Let $X = \Re^L_+$ be the commodity consumption set with typical element $x = (x_1, \ldots, x_L)$, and define $Y = X \times Z$ with typical element (x, z). The consumer is assumed to have a complete preference relation on Y, designated by $\succsim_Y$, which is typically represented by a utility function u_Y so that $u_Y(x^a, z^a) \geq u_Y(x^b, z^b) \iff (x^a, z^a) \succsim_Y (x^b, z^b)$ for all possible pairs of (x, z) vectors $(x^a, z^a), (x^b, z^b) \in Y$. I assume that $\succsim_Y$ is complete and transitive on Y, and also continuous, strictly convex and strongly monotone on X.

Usually the demand function is defined as the solution function to the constrained opti-

[1]The concept was first fully developed in Mäler (1971) and the terminology was introduced in Mäler (1974).

[2]Mäler's first example was fishery quality, although as a continuous variable.

mization problem,

$$\begin{aligned} \max_{x} \quad & u_Y(x,z) \\ s.t. \quad & p \cdot x \leq w, \\ & x \in X, \end{aligned} \tag{1}$$

where $p \in \Re^L_{++}$ is the vector of market good prices and $w > 0$ is the individual's wealth.[3] However the demand function can also be characterized in purely preference-theoretic terms,

$$\widehat{x}(p,z,w) = \{x \in X \,|\, p \cdot x \leq w, \text{ and } (x,z) \succsim_Y (\bar{x},z) \text{ for all } \bar{x} \text{ such that } p \cdot \bar{x} \leq w\}. \tag{2}$$

For each value of z, the consumer chooses the preference maximizing affordable commodity bundle so the nonmarket good essentially parameterizes the choice problem and the demand function (as do prices and wealth).

2 The problem of missing preference information

Suppose that there is a change in the nonmarket good "value" from z^a to z^b with price and wealth respectively fixed at $\bar{p}$ and $\bar{w}$, and we wish to measure the change in welfare. Typically this means finding some dollar value change in wealth that is either equivalent to the change in z or compensates for this change.[4] Working with the latter option, the change in welfare ΔW may be characterized either with preference notation or with the utility function,

$$\left(\widehat{x}\left(\bar{p},z^a,\bar{w}\right),z^a\right) \sim_Y \left(\widehat{x}\left(\bar{p},z^b,\bar{w}-\Delta W\right),z^b\right), \tag{3}$$

$$u_Y\left(\widehat{x}\left(\bar{p},z^a,\bar{w}\right),z^a\right) = u_Y\left(\widehat{x}\left(\bar{p},z^b,\bar{w}-\Delta W\right),z^b\right), \tag{4}$$

where $\sim_Y$ is the indifference relation associated with $\succsim_Y$.[5] With or without utility function representation, this requires preference information across different z values. That is, for

[3]Following Mas-Colell et al. (1995), I use wealth instead of income. The seminal Mäler (1971) uses "lump sum income" which is arguably more akin to wealth than income.

[4]These are equivalent variation and compensating variation respectively. See Mas-Colell et al. (1995) for the standard definition of these metrics, i.e., without nonmarket goods. Larson (1991), Bockstael and McConnell (1993), Herriges et al. (2004), Smith and Banzhaf (2004), and Bullock and Minot (2006) use a compensating variation measure for a change in the nonmarket good value that is the same as the one used here, while Ebert (1998) uses an equivalent variation measure.

[5]That is, $(x^a,z^a) \sim_Y (x^b,z^b) \Longleftrightarrow \left[(x^a,z^a) \succsim_Y (x^b,z^b) \text{ and } (x^b,z^b) \succsim_Y (x^a,z^a)\right]$ for all $(x^a,z^a),(x^b,z^b) \in Y$.

at least some pairs $(x^a, z^a), (x^b, z^b) \in Y$ with $z^a \neq z^b$ we need to know whether or not $(x^a, z^a) \succsim_Y (x^b, z^b)$. However, this information is not available from revealed preference.

Consumers reveal their preference as they choose different commodity bundles for various given combinations of p, z and w, as indicated by program (1) and equation (2). Since they are not able to choose z values in the market, revealed preference is not available for differences in these values. Furthermore, commodity revealed preference is only available across those bundles that the consumer might actually obtain as indicated by the demand function. For each $z \in Z$, the obtainable set in X is $\widehat{X}_z = \{x \in X \mid x = \widehat{x}(p, z, w) \text{ for some } (p, w) \in \Re^{L+1}_{++}\}$.[6] For each such z value, I assume that revealed preference provides a complete preference relation on $\widehat{X}_z$.[7] Thus for each $z \in Z$, we have a preference relation $\succsim_z$ such that for any $x^a, x^b \in \widehat{X}_z$ we know whether or not $x^a \succsim_z x^b$. From the properties of $\succsim_Y$ we also know that each $\succsim_z$ is transitive, continuous, strictly convex and strongly monotone. As part of this assumption there are also representative utility functions $u_z(x)$ such that $u_z(x^a) \geq u_z(x^b) \iff x^a \succsim_z x^b$.[8,9] Without loss of generality, I assume that these util-

[6]Three illustrative examples: 1) With Cobb-Douglas or CES preference, the obtainable set is the strictly positive orthant $\Re^L_{++}$ so that at least of some of each commodity is always consumed. 2) With Leontief preference the obtainable set is a ray from the origin (not including the origin). 3) With quasilinear preference, the obtainable set will typically include some commodity vectors with no consumption of some goods, i.e., corner solutions. This distinction between X and the obtainable subset is also used by Richter (1971).

[7]For example, these might be obtained by solving the integrability problem (see Mas-Colell et al. (1995)). This is the strongest assumption adopted in this paper. With it I have effectively assumed away all possible problems associated with recovering revealed preference for the standard situation with only market goods, i.e., without nonmarket goods. I have thereby narrowed the focus to those new problems that accompany the inclusion of nonmarket goods.

[8]The use of z as an index on $\succsim_z$ and $u_z(x)$ might suggest that Z is a countable set. However as I indicated previously, z may be a real valued continuous variable or even a real vector. In that context, $\succsim_z$ and $u_z(x)$ would probably be obtained as a parametric continuum.

[9]The individual u_z utility functions are generally not restrictions of the original unknown utility function $u_Y(x, z)$ to changes in x, as in $u_z(x) = u_Y(x, z)$. More precisely, we have no basis for knowing that they are, as otherwise we will have recovered $u_Y(x, z)$ and there would be no problem with nonmarket good welfare analysis. However, there is a relationship between these two types of utility functions (one unknown to the analyst and the other known). Let $z \in Z$. Then for any $x^a, x^b \in \widehat{X}_z$ we must have $\left[u_Y(x^a, z) \geq u_Y(x^b, z)\right] \iff \left[u_z(x^a) \geq u_z(x^b)\right]$. This can also be stated in terms of z-specific monotonic transformations. For each $z \in Z$ there is some real valued function f_z such that $f_z\left(u_Y(x, z)\right) = u_z(x)$ for all $x \in \widehat{X}_z$.

This discussion is illustrative of the complexity that is introduced with utility functions but not present with pure preference-theoretic work. All of the core findings of this paper could be developed and stated ity functions are continuously differentiable. The set $\{\succsim_z \mid z \in Z\}$ represents all available revealed preference.

From the set of all $\widehat{X}_z$ we can define the overall obtainable subset of Y, $\widehat{Y} = \{(x,z) \in Y \,|\, x \in \widehat{X}_z \text{ for } z \in Z\}$. I will use the notation $\succsim_{\widehat{Y}}$ to indicate the restriction of $\succsim_Y$ to the preference subdomain $\widehat{Y}$.[10] For welfare analysis we are only concerned with elements of Y that might actually occur with market interaction, i.e., those that can be obtained with the demand function. Thus at most, we are only interested in identifying the complete preference relation on $\widehat{Y}$, $\succsim_{\widehat{Y}}$. However what we have available is much less than this. For any given $z \in Z$ and any $x^a, x^b \in \widehat{X}_z$, we know whether or not $(x^a, z) \succsim_{\widehat{Y}} (x^b, z)$ since $x^a \succsim_z x^b \iff (x^a, z) \succsim_{\widehat{Y}} (x^b, z)$. However, for any $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$ with $z^a \neq z^b$, we cannot determine whether or not $(x^a, z^a) \succsim_{\widehat{Y}} (x^b, z^b)$. This preference information is not available from revealed preference. Thus our knowledge of preference on $\widehat{Y}$ is incomplete, and we are in particular missing preference information that is necessary to consider welfare issues involving distinctions in the value of the nonmarket good, such as finding the value of ΔW.

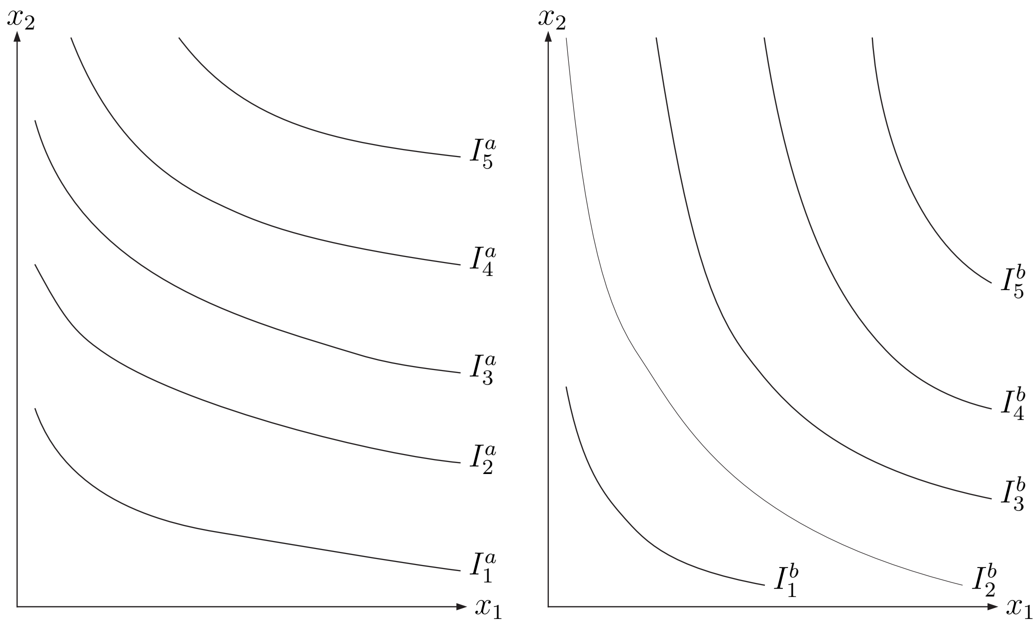
The missing preference information problem is illustrated by Figure 1, where $\succsim_{z^a}$ and $\succsim_{z^b}$ are depicted for some $z^a, z^b \in Z$ with $z^a \neq z^b$, $X = \Re^2_+$ and $\widehat{X}_{z^a} = \widehat{X}_{z^b} = \Re^2_{++}$. These two revealed preference relations are respectively represented by the I^a_i and I^b_i indifference curves. We know, for example, that all the points in I^a_5 are preferred to the points in I^a_3 and all the points in I^b_4 are preferred over the elements of I^b_1. However from revealed preference alone, we do not know whether or not the consumer prefers the points of I^a_5 (with $z = z^a$) over those of I^b_3 (with $z = z^b$). The problem is then identifying the remaining preference information that will enable us to compare the indifference curves in panel (a) with those in panel (b).

We can define a complete preference relation on $\widehat{Y}$ by specifying a complete one-to-one alignment between the z-specific sets of indifference surfaces that honors transitivity. For example with Figure 1, this might include associating I^a_i with I^b_i for all $i = 1, \ldots, 5$, or we might instead associate $I^a_1 \sim I^b_3$ and $I^a_2 \sim I^b_5$.[11] However, we cannot associate $I^a_1 \sim I^b_3$

more simply without any direct reference to utility functions. The most important role of utility functions here is that they permit comparison with a literature that is based on calculus methodology.

[10]Thus for any $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$, $(x^a, z^a) \succsim_{\widehat{Y}} (x^b, z^b) \iff (x^a, z^a) \succsim_Y (x^b, z^b)$.

[11]With these two examples we only have partial alignments, involving the depicted sampling of indifference curves, and hence are unable to specify a complete relation on $\widehat{Y}$ with $Z = \{z^a, z^b\}$.



(a) Revealed preference with $z = z^a$ (b) Revealed preference with $z = z^b$

Figure 1: Revealed preference with alternative nonmarket good values.

and $I_2^a \sim I_2^b$, as this would violate transitivity.[12] For each $z \in Z$, let $\mathcal{I}_z = \{I_i^z\}_{\forall i}$ be the set of all $\succsim_z$ indifference surfaces, and let $\mathcal{I} = \bigcup_z \mathcal{I}_z$ be the union of these sets. Then a complete one-to-one alignment between all $\mathcal{I}_z$ is a partition of $\mathcal{I}$, $\mathcal{P} = \{\mathcal{P}_j\}_{\forall j}$ such that each $\mathcal{P}_j$ includes exactly one indifference surface of $\mathcal{I}_z$ for each $z \in Z$.[13] If the partition also honors transitivity, then a preference relation on $\widehat{Y}$ can be constructed by treating each $\mathcal{P}_j$ as an equivalence class. In section 4 I use a complete one-to-one alignment based on weak complementarity to construct a complete preference relation on $\widehat{Y}$.

3 Weak complementarity properties

Weak complementarity is one method for providing the necessary preference information not available through revealed preference and thereby make welfare analysis possible. It is built on an relationship between the nonmarket good and one of the market goods. Mäler's first example was the relationship between the quality of a public fishery and sport fishing

[12] With the observed $I_2^a \succ I_1^a$ and $I_3^b \succ I_2^b$, the first association $I_1^a \sim I_3^b$, and transitivity, we obtain $I_2^a \succ I_2^b$ which contradicts the second association $I_2^a \sim I_2^b$.

[13] The i and j indexes of I_i^z and $\mathcal{P}_j$ do not imply that they are countable. They are always uncountable.

(nonmarket and market good respectively). Bockstael and McConnell (1993) provide two examples including wildlife populations in a sanctuary and trips to the sanctuary. In some presentations the nonmarket good is a quality of the market good, while others simply require that they be "consumed" together. Without loss of generality, I shall assume that the nonmarket good is associated with the first market good, x_1. Given this relationship, weak complementarity requires two preference properties, one which may be observed from revealed preference and another which is imposed as a preference assumption. This second property is thus a "preference restriction" or "maintained hypothesis." Neither property is consistently defined in the literature.[14]

The first property is nonessentiality. Willig (1978) originally defined it as requiring that "any bundle including good 1 can be matched in the preference ordering by some other bundle which excludes good 1."[15] Formally, for any $x^a \in X$ and $z \in Z$ there must exist some $x^b = (x_1^b, \ldots, x_L^b) \in X$ with $x_1^b = 0$ such that $(x^a, z) \sim_Y (x^b, z)$, or in terms of utility, $u_Y(x^a, z) = u_Y(x^b, z)$. I shall call this property weak nonessentiality.

Most of the weak complementarity literature instead requires a stronger property that I shall also adopt.[16] Modifying Willig's definition, with strong nonessentiality, "any *demanded* bundle including good 1 can be matched in the preference ordering by some *demanded* bundle which excludes good 1." Thus the distinction is that we are restricting ourselves to those commodity vectors that can be obtained via the demand function. Holding wealth constant, this requires that for any $p^a \in \Re^L_{++}$, $w > 0$ and $z \in Z$ there must exist some $p^b \in \Re^L_{++}$ such that with the first component demand function $\widehat{x}_1(p^b, z, w) = 0$ and $(\widehat{x}(p^a, z, w), z) \sim_Y \big(\widehat{x}(p^b, z, w), z\big)$. With our previously defined obtainable sets we can state this more simply as requiring that for any $z \in Z$ and any $x^a \in \widehat{X}_z$, there must exist some $x^b \in \widehat{X}_z$ with $x_1^b = 0$

[14] Bullock and Minot (2006) provide an interesting analysis of alternative definitions of weak complementarity in terms of the implied path of integration.

[15] The word "other" is not actually operational in the definition so that the two bundles can be the same. von Haefen (2007) provides an equivalent characterization of nonessentiality.

[16] The literature actually tends to be quite vague on nonessentiality. It is sometimes not explicitly considered while clearly still an implicit requirement such as with Larson (1991), Herriges et al. (2004), and Bullock and Minot (2006). Others, such as Smith and Banzhaf (2004) initially state it in terms Willig's original characterization but in their analysis clearly require the stronger version adopted here. My statement of strong nonessentiality is equivalent with the characterizations provided by Bockstael and McConnell (1993), and Palmquist (2005).

such that $(x^a, z) \sim_Y (x^b, z)$.[17] We then also have $(x^a, z) \sim_{\widehat{Y}} (x^b, z)$ and $x^a \sim_z x^b$.

With weak nonessentiality but without strong nonessentiality, intersection points such as x^b are not members of the obtainable sets, $\widehat{X}_z$. Both weak and strong nonessentiality can be directly observed with revealed preference. The distinctions between the two characterization of nonessentiality are illustrated in Figure 2. First in panel (a), the indifference curves do not touch the vertical axis so that neither characterization is satisfied. In panel (b) the indifference curves touch the vertical axis and obtain tangents there.[18] This satisfies Willig's definition but not the more widely used definition of nonessentiality as it not possible to obtain any of these points as demand.[19] Finally in panel (c) the indifference curves intersect the vertical axis with finite slopes. These points can be obtained with positive real prices, satisfying both weak and strong nonessentiality. With wealth and other prices fixed, the choke price for good one is the minimal price for which the consumption of good one is zero. The three panels of Figure 2 respectively illustrate no choke prices, infinite choke prices and finite choke prices. Choke prices are discussed further in section 5.

No existence value is the second required preference property for using the weak complementarity method for nonmarket good welfare analysis.[20] This property tells us that the consumer does not care about the value of z when the consumption bundle is fixed with $x_1 = 0$. Thus the nonmarket good does not have any stand-alone existence value, but rather has only use value in conjunction with the consumption of good one.[21] Beginning with Mäler (1971), this property is typically formally defined by the partial differential equation,

$$\frac{\partial u_Y}{\partial z}(0, x_2, x_3, \ldots, x_L, z) = 0. \tag{5}$$

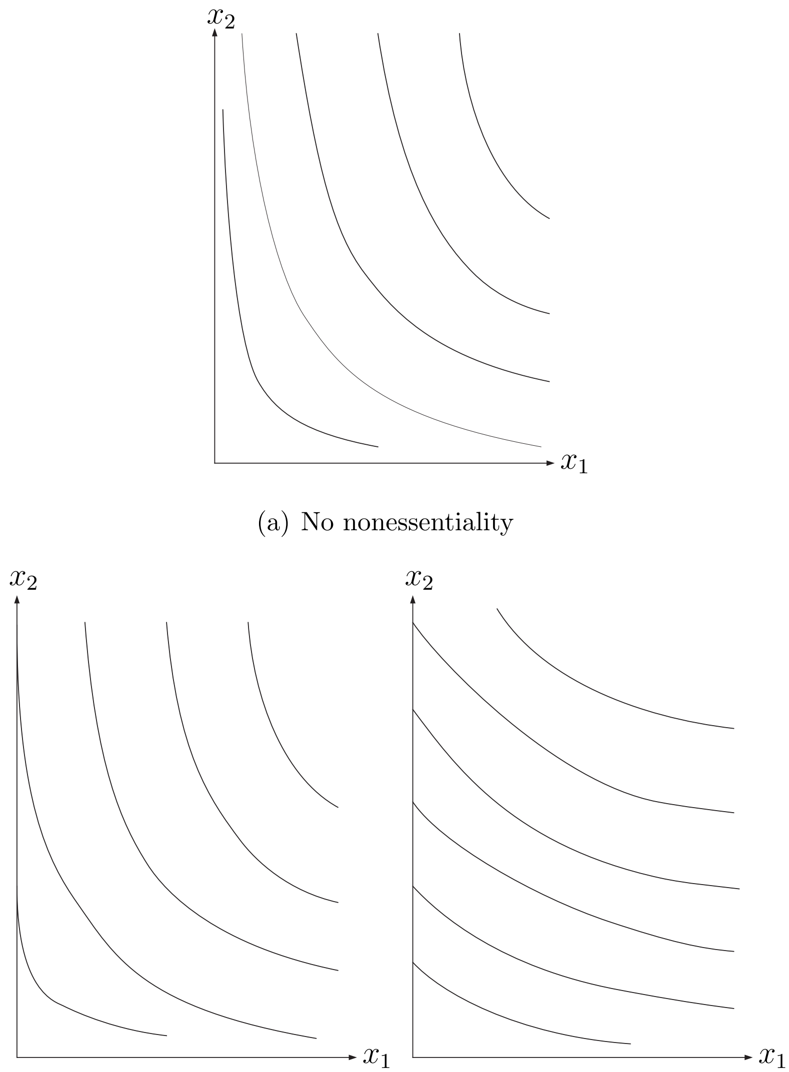
[17] A conjecture: Let X_1^0 be the subset of X where $x_1 = 0$, $X_1^0 = \{x \in X \,|\, x_1 = 0\}$. Then I believe that strong nonessentiality is equivalent to the requirement that $X_1^0 \backslash \{0\} \subset \widehat{X}_z$ for all $z \in Z$.

[18] Depending on how tangency is defined, these might not be considered tangents since the indifference curves also end at these points. However the intuition of a common slope is preserved. Tangents like this may occur with CES preference.

[19] This is a consequence of our requirement for positive real valued prices. Demand could only be obtained at these points of tangency with the vertical axis if the price of good one were infinite or the price of good two were zero.

[20] Beginning with Mäler (1974), weak complementarity is defined in most of the literature as the property that I call no existence value. However the need for both properties is often not clear in these presentations. My two-property definition of weak complementarity follows Palmquist (2005) and von Haefen (2007), and facilitates a more clear understanding of the distinct roles of both properties.

[21] See Herriges et al. (2004) for a more precise understanding of use and existence value.



(a) No nonessentiality

(b) Weak nonessentiality only

(c) Strong nonessentiality

Figure 2: Types of Nonessentiality.

For our purposes there are two problems with this formal characterization.[22] The first is technical in that the equation has no meaning if z is not a continuous real variable, severely limiting the range of possible nonmarket goods as described in the first section. The second problem is conceptual. Explicitly, equation (5) is only a first order marginal characterization that looks like any one of the first order optimality conditions that are so ubiquitous in neoclassical microeconomics. However no existence value is not about a marginal utility

[22] The no existence value property is also sometimes characterized in terms of marginal indirect utility and marginal expenditure (for example, Bradford and Hildebrandt (1977), Bockstael and McConnell (1993), and Smith and Banzhaf (2004)). These characterizations also suffer from these two problems.

condition and is also not about optimality. Equation (5) does not directly convey the key intuition of no existence value that the consumer is indifferent across all values of z.

As consequence of these two problems I instead primarily use a preference characterization of no existence value: for any $x \in X$ with $x_1 = 0$ and any $z^a, z^b \in Z$ we have $(x, z^a) \sim_Y (x, z^b)$. The equivalent utility characterization requires that $u_Y(x, z^a) = u_Y(x, z^b)$.[23] However it is technically characterized, it is clear that no existence value cannot be verified from revealed preference and is hence an assumption, that is a "preference restriction" or "maintained hypothesis."

4 Exact welfare measurement

In this section I use the two weak complementarity preference properties to obtain an exact welfare measure based on equations (3) and (4) by way of preference construction. For now I only the consider the case where $L = 2$, so that in addition to the weak complement, good one, we have good two that is usually described as a composite good.[24] This restriction permits more a intuitive presentation. However not all aspects of this development directly generalize to longer commodity vectors. I deal with that issue at the end of section 6.

As discussed in section 2, the methodological problem of nonmarket good welfare analysis stems from incomplete preference information across $\widehat{Y}$. I will now construct a complete preference relation on $\widehat{Y}$ using revealed preference and the two preference properties of weak complementarity. This construction is based on fanned indifference curves graphical analysis as developed by Smith and Banzhaf (2004) and presented in my Figure 3. This approach involves overlaying indifference curves for multiple values of z in the same graph. Two indifference curves for each of two z values are depicted in Figure 3, I_1^a & I_2^a for $z = z^a$ and I_1^b & I_2^b for $z = z^b$. These and other indifference curves would be known from available revealed preference, $\{\succsim_z \mid z \in Z\}$. Strong nonessentiality is indicated by all the indifference curves intersecting the vertical axis with finite slopes. A "fan" is defined as the collection all indifference curves that intersect the vertical axis at the same point. I shall use the term "handle" to refer to the point where an individual indifference curve intersects the vertical axis, and hence the term "fan handle" for the intersection point of a fan of all the curves

[23] Willig (1978) also uses this utility equality characterization.

[24] Smith and Banzhaf (2004), and Palmquist (2005) also couch their development in this two good context.

that share the same handle. Two representative fans are depicted in Figure 3, each with two representative curves, located respectively at the fan handles defined by the good two values x_2^1 and x_2^2. Actually, there are fan handles located at each positive value along the vertical axis and each fan includes an indifference curve for each value of z, which for most applications is infinite.

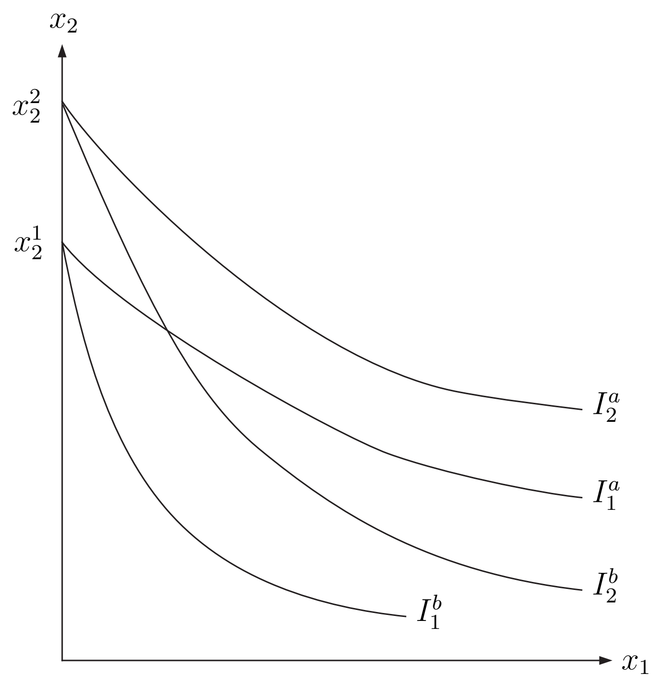


Figure 3: Fanning indifference curves for weak complementarity.

The no existence value assumption tells us that the consumer is indifferent between indifference curves belonging to the same fan, so that each fan defines an indifference set across both x and z values. Since each $(x, z) \in \widehat{Y}$ belongs to a unique z-indifference curve which in turn belongs to a unique fan, we have fully partitioned $\widehat{Y}$ into a set of indifference sets. Monotonicity imposes a strict preference ordering on the fan handles that increases with the x_2 value. This ordering carries over to the fans themselves so that we have a complete and transitive preference relation on $\widehat{Y}$ designated by $\succsim_{wc}$.[25] In terms of the discussion at the

[25]This weak complementarity preference relation may be precisely defined in terms revealed preference. Let $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$. With nonessentiality there exists unique $(0, x_2^\alpha) \in \widehat{X}_{z^a}$ and $(0, x_2^\beta) \in \widehat{X}_{z^b}$ such that $(0, x_2^\alpha) \sim_{z^a} (x^a)$ and $(0, x_2^\beta) \sim_{z^b} (x^b)$. From strong monotonicity of $\succsim_{z^a}$ we have $(0, x_2^\alpha) \succsim_{z^a} (0, x_2^\beta) \iff x_2^\alpha \geq x_2^\beta$, and hence from transitivity of $\succsim_{z^a}$, have $x^a \succsim_{z^a} (0, x_2^\beta) \iff x_2^\alpha \geq x_2^\beta$. Since the weak complementarity preference relation must be consistent with revealed preference, we have $\big((0, x_2^\beta), z^b\big) \sim_{wc} (x^b, z^b)$ and $(x^a, z^a) \succsim_{wc} \big((0, x_2^\beta), z^a\big) \iff x_2^\alpha \geq x_2^\beta$. With no existence value, $\big((0, x_2^\beta), z^a\big) \sim_{wc} \big((0, x_2^\beta), z^b\big)$. The

very end of section 2, the construction of $\succsim_{wc}$ is based on a complete one-to-one alignment between the $\mathcal{I}_z$ where the fans define the partition of $\mathcal{I}$. As required, each $\mathcal{P}_j$ (each fan) includes exactly one indifference surface of $\succsim_z$ for each $z \in Z$. The relation $\succsim_{wc}$ is well defined and hence unique. However since its construction depends on the no existence value assumption, we do not know if it is the same as the unknown original relation $\succsim_{\widehat{Y}}$.

Following equation (3), given a change in the nonmarket good value from z^a to z^b (with $z^a, z^b \in Z$), a compensating measure of the change in welfare based on weak complementarity requires for any $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \Re^2_{++}$ and $\bar{w} > 0$, that we be able to produce some real number ΔW such that,

$$\left(\widehat{x}\left(\bar{p}, z^a, \bar{w}\right), z^a\right) \quad \sim_{wc} \quad \left(\widehat{x}\left(\bar{p}, z^b, \bar{w} - \Delta W\right), z^b\right). \tag{6}$$

I provide a graphical geometric procedure here for specifying ΔW followed by a parallel computational process using the u_z utility functions. With equation (6), we have an equivalence statement in the context of weak complementarity and are therefore working with the single fan that includes the point $x^a = \widehat{x}\left(\bar{p}, z^a, \bar{w}\right)$, as depicted in Figure 4. All we need to do is plot tangents lines with slope $-\bar{p}_1/\bar{p}_2$ to both the z^a and z^b indifference curves of that fan. The vertical distance between these two lines is $|\Delta W/\bar{p}_2|$ and the horizontal distance is $|\Delta W/\bar{p}_1|$. ΔW is positive if the z^a tangent line is above the other and negative otherwise.

ΔW can be calculated with the u_z utility functions by following these steps that emulate the preceding graphical approach: 1) starting with $x^a = (x_1^a, x_2^a)$, calculate the utility value $\bar{u}_a = u_{z^a}(x^a)$; 2) solve the equation $u_{z^a}(0, x_2^h) = \bar{u}_a$ for the handle value x_2^h; 3) find the z^b utility value at the handle, $\bar{u}_b = u_{z^b}(0, x_2^h)$; 4) find the z^b-Hicksian demand for $\bar{p}$ and $\bar{u}_b$, $x^b = (x_1^b, x_2^b)$ by solving the system of two equations, $u_{z^b}(x^b) = \bar{u}_b$ and

$$\frac{\dfrac{\partial u_{z^b}}{\partial x_1}(x^b)}{\dfrac{\partial u_{z^b}}{\partial x_2}(x^b)} = \frac{\bar{p}_1}{\bar{p}_2};$$

and finally 5) calculate $\Delta W = \bar{p}_1(x_1^a - x_1^b) + \bar{p}_2(x_2^a - x_2^b)$. Aspects of this construction are also illustrated in Figure 4.

For many applications it is useful to have a utility function that represents $\succsim_{wc}$ on $\widehat{Y}$. I next develop a utility function that is calibrated by the handle values. With strong

transitivity of $\succsim_{wc}$ follows from the transitivity of the relation $\geq$ on the quantity x_2. With this transitivity we obtain $(x^a, z^a) \succsim_{wc} (x^b, z^b) \iff x_2^\alpha \geq x_2^\beta$ so that $\succsim_{wc}$ is complete. This construction works with both $z^a = z^b$ and $z^a \neq z^b$ (as well as with both $x^a = x^b$ and $x^a \neq x^b$).

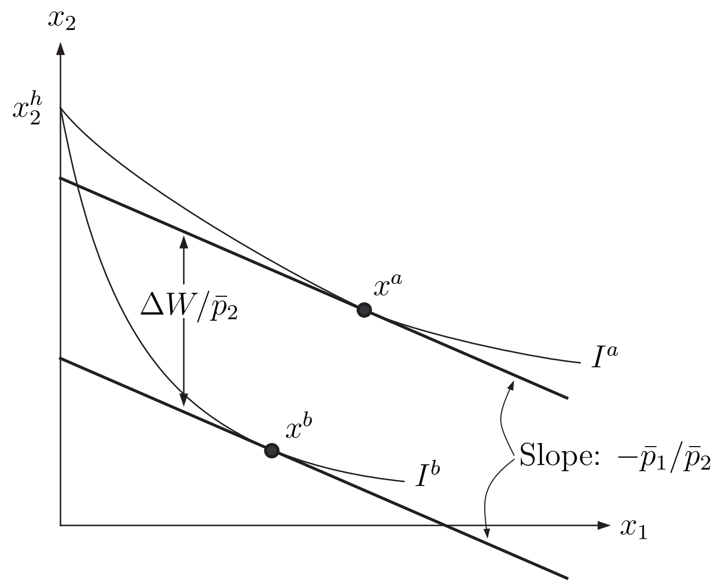


Figure 4: Construction of exact welfare measure with weak complementarity.

nonessentiality and strong monotonicity, for any $(x, z) \in \widehat{Y}$ there must exist a unique quantity of good two, x_2^h, such that $u_z(x) = u_z(0, x_2^h)$. I define the weak complementarity utility value of (x, z) to be x_2^h, $u_{wc}(x, z) = x_2^h$. Thus for all $(x, z) \in \widehat{Y}$,

$$u_{wc}(x, z) = \left\{ x_2^h > 0 \,|\, u_z(x) = u_z(0, x_2^h) \right\}.$$

With this function, all (x, z) elements on the same fan have the same utility value and higher utility values are assigned to those fans with larger handle values. It is clear that u_{wc} represents $\succsim_{wc}$ on $\widehat{Y}$ and is unique up to a monotonic transformation.[26] With it and the demand function we can construct an indirect utility function $v_{wc}(p, z, w) = u_{wc}(\widehat{x}(p, z, w), z)$, by solving $u = v_{wc}(p, z, w)$ for w we can obtain the expenditure function $w = e_{wc}(p, z, u)$, and we construct the Hicksian demand function $h_{wc}(p, z, u) = \widehat{x}(p, z, e_{wc}(p, z, u))$. The construction of u_{wc}, v_{wc}, e_{wc} and h_{wc} does not require any calculus except for initially determining the individual u_z functions, and therefore works equally well with discrete and continuous nonmarket goods. Again without resorting to calculus, these constructions can be used for exact welfare analysis. For $\bar{p} \in \Re^2_{++}$, $\bar{w} > 0$ and $z^a, z^b \in Z$, the weak complementarity compensating variation for a change from z^a to z^b is $\Delta W = \bar{w} - e_{wc}\left(\bar{p}, z^b, \hat{u}_a\right)$ where $\hat{u}_a = v_{wc}(\bar{p}, z^a, \bar{w})$.

[26]For each $z \in Z$, u_{wc} is also a monotonic transformation of u_z so that there exists some z-specific real valued function f_z such that $u_{wc}(x, z) = f_z(u_z(x))$ for all $x \in \widehat{X}_z$.

5 Unnecessary assumptions

In this section I discuss two other assumptions that are often adopted in the literature in conjunction with weak complementarity, and also consider doing without strong nonessentiality. Part of my message is that we can obtain an exact welfare measure without these assumptions, indicating that a preference-theoretic approach may be more parsimonious in its need for additional information and hence more efficient in its use of the available preference information in the form of revealed preference. However this literature generally does not explicitly utilize my strongest assumption, that we can recover revealed preference in the form of all the z-fixed relations $\{\succsim_z \mid z \in Z\}$ and their respective representative utility functions. This literature does universally assume the availability of complete individual Marshallian demand functions of the form $\widehat{x}(p, z, w)$, and also includes much discussion of what can and cannot be known from revealed preference but without actually specifying revealed preference. This paper provides an explicit examination of what can be done with the revealed preference available from Marshallian demand and the two defining properties of weak complementarity.

The most pervasive and crucial additional assumption is that the nonmarket good is a continuous real variable and that demand and utility are both differentiable with respect to it.[27] The continuity assumption by itself severely limits the range of possible applications, excluding discrete nonmarket goods as well as continuousness nonmarket goods with incomplete data. Requiring continuous information for the analysis of a discrete change seems to be superfluous. Imposing differentially on top of this may further restrict the range of applications. The current theoretical literature on weak complementarity is dominated by a discussion on the use and possible necessity of invoking the Willig condition to obtain exact welfare measures.[28] Originally presented in Willig (1978), this is actually three equivalent conditions concerning the partial derivatives of Marshallian demand and indirect utility with respect to z. Consequently, this discussion cannot have any relevance to discrete nonmarket goods. Furthermore, the Willig condition is not observable from revealed preference so that invoking it imposes an additional assumption. It is true that continuity and differentiability

[27] With the exception of Herriges et al. (2004), all of the nonmarket goods literature cited in this article relies on this assumption for their core developments.

[28] This discussion includes Bockstael and McConnell (1993), Smith and Banzhaf (2004), Palmquist (2005), Bullock and Minot (2006) and von Haefen (2007). Bullock and Minot (2006) show that the Willig condition is not necessary for exact welfare analysis.

are necessary for some welfare measures such as various marginal willingness to pay metrics. With those assumptions, such measures can still be developed with the u_{wc}, v_{wc}, e_{wc} and h_{wc} functions as specified here. Nothing presented here precludes the use of calculus techniques.

A monotonic preference on the nonmarket good is often assumed in the context of weak complementarity.[29] Although typically specified as a partial derivative inequality such that u_Y strictly increases in value with z, this "goodness" assumption can be equivalently defined as requiring for all $z^a, z^b \in Z$ where $Z \subseteq \Re$ that $z^b > z^a \Rightarrow (x, z^b) \succ_Y (x, z^a)$ for all $x \in X$.[30] However this violates no existence value for those points in X where $x_1 = 0$.[31] I will therefore restrict the goodness assumption to those points in X where $x_1 \neq 0$. The fans presented in both Figures 3 and 4 illustrate this restricted notion of goodness. For each of these three fans, the I^a indifference curve is above the I^b curve to the right of the vertical axis, indicating that z^b is preferred over z^a with weak complementarity.[32] If with all fans of a given application, the I^a curve is always above the I^b curve, then z^b is universally regarded as better than z^a (i.e., for all values of $x \in X$ with $x_1 \neq 0$).

However goodness is not necessary for implementing weak complementarity. For example consider the twisted fans presented in Figure 5. Goodness is violated with z^b preferred over z^a to the left of the dashed line and z^a preferred over z^b to the right. This lack of goodness has no effect on the constructions presented in the proceeding section. In particular, we can still define the weak complementarity preference relation by treating each fan as an indifference set in $\widehat{Y}$ and ranking the fans by handle value. Moreover, we can do this no matter how tangled and knotted the fans may be. Most of the weak complementarity examples in the literature, such as Mäler's fishery quality, invoke an intuition for goodness. However goodness is not necessary for implementing weak complementarity, broadening the range of possible applications.

As defined here, revealed preference is restricted to the $\widehat{X}_z$ sets. Consequently with only

[29] Bockstael and McConnell (1993), Neill (1999), Herriges et al. (2004), Smith and Banzhaf (2004), and von Haefen (2007) impose this assumption with weak complementarity. Ebert (1998) uses it in a different context.

[30] See Brown (2008) for a more in depth preference-theoretic consideration of goodness.

[31] None of the papers cited in footnote 29 indicate awareness of this technical contradiction.

[32] Let x^α be any point on the I^a curve of one of these fans (with $x_1^\alpha \neq 0$) and let x^β be the point on the I^b curve just below it, so that $x_1^\alpha = x_1^\beta$. Since they are on the same fan, $x^\alpha \sim_{wc} x^\beta$. With strong monotonicity, it follows that z^b is preferred over z^a in order to compensate for the smaller amount of good two and thereby achieve indifference.

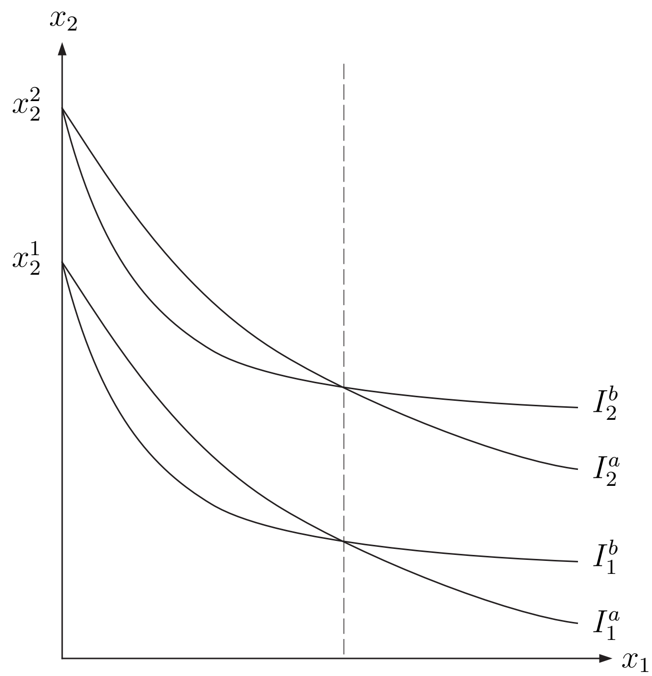


Figure 5: Simple tangled fans.

weak nonessentiality, revealed preference does not include that portion of X where $x_1 = 0$, and it is not possible to construct $\succsim_{wc}$ as we have from all of the $\succsim_z$. However we can extend the revealed preference domain for each $z \in Z$ to include the vertical axis.

For any fixed $z \in Z$, let I^z_α and I^z_β be any two distinct indifference curves defined by $\succsim_z$. Without loss of generality we may assume that I^z_α is superior to I^z_β. All the points in both curves are elements of $\widehat{X}_z$. Therefore with only weak nonessentiality, generally neither of these revealed indifference curves will include a point on the vertical axis (the curves end just before the axis much like the end of an open line segment). Let x_1^1 be a positive value of good one that is included in both indifference sets.[33] Then with strong monotonicity, all good one quantities $x_1 \in (0, x_1^1]$ are also included in both indifference curves. Let $\{x_1^n\}_{n=1}^\infty$ be a decreasing sequence with elements from $(0, x_1^1]$ such that $\lim_{n\to\infty} x_1^n = 0$. For each x_1^n in the sequence, let $x_{2\alpha}^n$ and $x_{2\beta}^n$ be the respective values of good two such that $(x_1^n, x_{2\alpha}^n) \in I^z_\alpha$ and $(x_1^n, x_{2\beta}^n) \in I^z_\beta$. Then for all $n = 1, \ldots, \infty$, $(x_1^n, x_{2\alpha}^n) \succ_z (x_1^n, x_{2\beta}^n)$ and hence also $((x_1^n, x_{2\alpha}^n), z) \succ_{\widehat{Y}} \big((x_1^n, x_{2\beta}^n), z\big)$. Let $x^\alpha = \lim_{n\to\infty}(x_1^n, x_{2\alpha}^n)$ and $x^\beta = \lim_{n\to\infty}(x_1^n, x_{2\beta}^n)$. With weak nonessentiality, these two points are well defined as they are the required intersections of the respective original $\succsim_Y$ indifference curves with the vertical axis, such as depicted in

[33]It is possible that not all positive quantities are included, such as with quasilinear preference.

panel (b) of Figure 2. Then since $\succsim_Y$ is continuous on X, we have $(x^\alpha, z) \succ_Y \left(x^\beta, z\right)$. We have thus extended all of the revealed preference indifference curves to includes points on the vertical axis.

Thus with weak nonessentiality we are able to recover the $\succsim_Y$ preference over all of X for each fixed value of z, just as we did previously with strong nonessentiality and revealed preference on each $\widehat{X}_z$. We can then use the same general construction techniques for specifying a weak complementarity preference relation and exact welfare measures using the fans as indifference sets. The limit points obtained above are the handles of those fans. The distinction between weak and strong nonessentiality is generally discussed in the literature in terms of infinite versus finite choke prices. Recently Bullock and Minot (2006) have shown that it is not possible to implement weak complementarity with infinite choke prices (weak nonessentiality) using welfare measures defined by paths of integration in (p, z) space. Thus with a preference-theoretic methodology we are again able to go beyond what is possible with standard calculus techniques.

6 Testing no existence value

For the proceeding two sections there were only two market goods. When there are more than two goods it is possible for revealed preference to be inconsistent with no existence value so that weak complementarity is not feasible. In this section I show this and provide generalizations of the welfare constructions presented in section 4.

It is well understood that there are an infinite number of diverse preference relations on $\widehat{Y}$ that are consistent with revealed preference.[34] It follows that there can be no test on revealed preference for verifying that the "true" relation $\succsim_{\widehat{Y}}$ satisfies no existence value. Many authors have generalized this to state that there can be no test at all, implying that we cannot use revealed preference to determine in any way whether or not the relation satisfies no existence value.[35] However this generalization is fallacious. The inability to affirm that

[34]Herriges et al. (2004) and von Haefen (2007) provide a clear statements of this in terms of utility function transformations.

[35]For example Bockstael and McConnell (1993, p. 1256) conclude that "weak complementarity is an untestable hypothesis," Herriges et al. (2004, p. 56) describe the choice between alternative welfare measures as "the choice between non-testable preference restrictions," and von Haefen (2007, p. 16) characterizes weak complementarity as an "intuitive but untestable restriction." Ebert (2001, p. 374) states that "one is unable

$\succsim_{\widehat{Y}}$ satisfies no existence value does imply an inability to deny it. We can use revealed preference to deny or falsify the possibility of no existence value. No existence value cannot be true for any of the many feasible preference relations on $\widehat{Y}$ when it is inconsistent with revealed preference.

In section 4, with $L = 2$ and observed strong nonessentiality, we specified a weak complementarity preference relation that could not be falsified by revealed preference. Therefore falsification is only possible with a larger number of market goods. I shall begin by working with $L = 3$ where it is still possible to use graphical imagery and geometric intuition. As we go from two to three market goods, keeping good one as the weak complement, all of the graphical components increase by one dimension. For example, previously indifference sets were simple curves but are now surfaces. With strict convexity, these surfaces would look something like sails in a full wind. The important region of X defined by $x_1 = 0$, $X_1^0 = \{x \in X \,|\, x_1 = 0\}$, previously the vertical axis, is now a quarter plane taking on all nonnegative combinations of the other two goods. Previously the indifference sets intersected with this region at single points. Now with curved surfaces intersecting a plane, these indifference set handles are curves.

This is depicted in Figure 6 with two such curves, one indifference surface handle each for two nonmarket good values, $z^a, z^b \in Z$. These curves would be observable with strong nonessentiality and our definition of revealed preference.[36] Suppose that two handles cross as depicted at a single point x^1. Then there must be a point such as x^2 that is on one of these two curves but not the other. In this case, from revealed preference we have $x^1 \sim_{z^a} x^2 \;\Rightarrow\; (x^1, z^a) \sim_{\widehat{Y}} (x^2, z^a)$. Now suppose that the original preference relation on $\widehat{Y}$ satisfies no existence value so that we also have $(x^1, z^a) \sim_{\widehat{Y}} (x^1, z^b)$ and $(x^2, z^a) \sim_{\widehat{Y}} (x^2, z^b)$. Then by transitivity we get $(x^1, z^b) \sim_{\widehat{Y}} (x^2, z^b)$ and hence $x^1 \sim_{z^b} x^2$ which directly contradicts the observed defining characteristic of x^2. Thus with the revealed preference scenario depicted in Figure 6, we are able to refute or falsify the no existence value preference restriction.

Ebert (1998, p. 242) states that preference restrictions are untestable in general because "there are no observable implications of the properties imposed." However there can be observable implications and from Figure 6 we can discern the observable implications of weak complementarity. Falsification of no existence value is possible with this revealed

to reject" preference restrictions.

[36]Or with weak nonessentiality and an extended revealed preference using continuity, such as developed at the end of section 5.

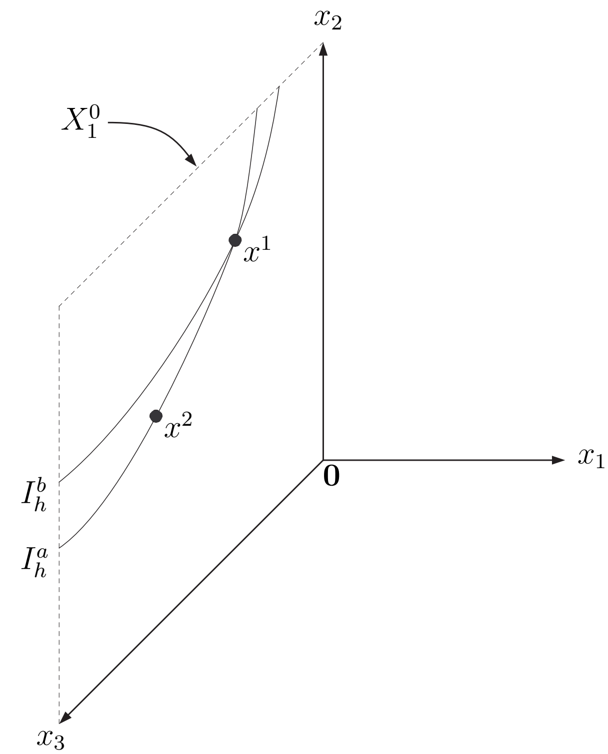


Figure 6: Falsifying no existence value with revealed preference

preference scenario because the z^a indifference set handle including x^1 is not coincident with the z^b indifference set handle that also includes x^1. With general L, strong nonessentiality, no existence value and any $z^a, z^b \in Z$, let I_h^a be a z^a indifference set handle. Thus $I_h^a \subset X_1^0$ and $x^1 \sim_{z^a} x^2$ for all $x^1, x^2 \in I_h^a$. Let $x^1, x^2 \in I_h^a$ and let I_h^b be the z^b indifference set handle that includes x^1. Then as before, from no existence value and transitivity of $\succsim_{\widehat{Y}}$ we have $x^1 \sim_{z^b} x^2$ so that $x^2 \in I_h^b$ and hence $I_h^a = I_h^b$. Thus the set of indifference set handles is invariant across z values. I call this condition "single-preference on X_1^0." No existence value is not feasible without single-preference. This is a trivial consequence when $L = 2$ and all indifference set handles are single points, but not trivial when $L \geq 3$. With strong nonessentiality, single-preference is observable from revealed preference. We thus have an observable consequence from imposing no existence value. Stated in the other direction, when $L \geq 3$ the absence of single-preference is a testable condition that allows us to reject no existence value and with it weak complementarity.

If revealed preference exhibits single-preference on X_1^0, we can generalize the welfare constructions developed in section 4 for general L. We constructed $\succsim_{wc}$ by associating all the observable indifference sets with a common handle. These super indifference sets are the fans of Smith and Banzhaf. They are well defined with general L and single-preference on X_1^0. Strong monotonicity imposes a strict ordering on the fan handles so that $\succsim_{wc}$ is well defined on $\widehat{Y}$. With monotonicity each fan handle includes a single point on the "45^o line" in X_1^0. These points are of the form $(0, x_2^h, \ldots, x_L^h)$ where $x_2^h = x_3^h = \ldots = x_L^h$. We can generalize the utility function u_{wc} that represents $\succsim_{wc}$ by assigning this common commodity value to all (x, z) elements of the fan. The v_{wc}, e_{wc} and h_{wc} functions can then be defined as before. The geometric procedure for specifying ΔW is generalized by using tangent planes when $L = 3$ and tangent hyperplanes when $L \geq 4$.

7 Conclusions

The weak complementarity methodological literature is almost entirely based on constructions in differential and integral calculus.[37] Preference-theoretic characterizations and constructions, such as those provided here, offer an alternative methodological approach that can supplement calculus-based methods. In particular, our preference-theoretic characterization of weak complementarity allows us to directly construct a welfare measure based on an explicit specification of revealed preference. This provides a clarity in our understanding of the role of revealed preference that is not available with the usual calculus-based methodologies. Over reliance on calculus methodology also seems to foster carelessness in the characterization of some preference properties such as with nonessentiality. The rigor and intrinsic transparency required by direct preference construction mitigates against this lack of precision, and can also yield more intuitive and direct characterizations such as with no existence value. Our exact welfare measure requires neither a continuous nonmarket good, nor monotonic preference on the nonmarket good, demonstrating that the preference-theoretic approach can be more frugal in the use of preference information and assumptions. Furthermore, this measure can be easily extended to accommodate infinite choke prices, which is not possible with current integral definitions of welfare. Thus weak complementarity can be applied in many contexts that are not feasible with pure calculus constructions. Finally, while several authors working in the realm of calculus have indicated it was impossible, I have obtained a nontrivial observable condition that is necessary for the feasibility of no existence value. The presence or absence of this single-preference condition in revealed preference is thus a negative test of no existence value, and with it, weak complementarity.

[37] One exception is Herriges et al. (2004).

Taken altogether, this preference-theoretic approach allows us to do more with less.

I do not advocate completely replacing calculus with preference-theoretic modeling. Each of these methodologies has its advantages. Calculus is an exact tool for understanding the consequences of infinitesimal changes and also allows us to take advantage of the special properties of continuous and differentiable functions. However, it is typically not a precise tool for understanding information content as with multiple sources of preference information (revealed and assumed). Combining preference-theoretic and calculus-based methodologies would allow the researcher to distinguish between, on the one hand, revealed preference conditions necessary for the feasibility of a preference restriction, and on the other hand, analytic properties that are most convenient for actually specifying an applicable welfare measure. Nevertheless, the utility of calculus is substantially diminished in some circumstances such as with discrete nonmarket goods and infinite choke prices.

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