E. J. OURSO COLLEGE OF BUSINESS
Department of Economics

DEPARTMENT OF ECONOMICS WORKING PAPER SERIES

Modeling Resource Flow Asymmetries using Condensation Networks

Pascal Billand
Universite′ de Lyon, Universite′ Jean Monnet &
CNRS, GATE Lyon St. Etienne

Christophe Bravard
Universite′ de Lyon, Universite′ Jean Monnet &
CNRS, GATE Lyon St. Etienne

Sudipta Sarangi
DIW Berlin & Louisiana State University

Working Paper 2012-01
http://bus.lsu.edu/McMillin/Working_Papers/pap12_01.pdf

Department of Economics
Louisiana State University
Baton Rouge, LA 70803-6306
http://www.bus.lsu.edu/economics/

Modeling Resource Flow Asymmetries using Condensation Networks

PASCAL BILLAND[a], CHRISTOPHE BRAVARD[a], SUDIPTA SARANGI[b]

[a] *Université de Lyon, Lyon, F-69003, France ; Université Jean Monnet, Saint-Etienne, F-42000, France ; CNRS, GATE Lyon St Etienne, Saint-Etienne, F-42000, France. email: pascal.billand@univ-st-etienne.fr, christophe.bravard@univ-st-etienne.fr*

[b] *DIW Berlin and Department of Economics, Louisiana State University, Baton Rouge, LA 70803, USA. email: sarangi@lsu.edu*

Abstract

This paper analyzes the asymmetries with regard to the resources obtained by groups of players in equilibrium networks. We use the notion of *condensation networks* which allows us to partition the population into sets of players who obtain the same resources and we order these sets according to the resources obtained. We establish that the nature of heterogeneity plays a crucial role on asymmetries observed in equilibrium networks. Our approach is illustrated by introducing the *partner heterogeneity* assumption into the one-way flow model of Bala and Goyal [1].

JEL Classification: C72, D85

Keywords: Nash networks, one-way flow model, condensation networks, chain, inf-semi-lattice.

1 Introduction

Heterogeneity and asymmetries are common features of networks in the real world. In his recent book, Jackson [2] argues that theoretical models need to incorporate heterogeneity as a natural extension. Heterogeneity in such models takes the form of players having different values and costs associated with them. Similarly, asymmetries between players positions (see for instance Newman [3] or Jackson [2]) may play a fundamental role in determining outcomes for players with regard to the resources obtained and differences in economic performance. Such heterogeneity and asymmetries do however impose a cost − they tend to increase the set of equilibrium networks. As a result, the characterization of equilibrium networks can become quite complicated. In this paper we provide an alternative approach to characterize equilibrium networks using the notion of condensation networks. The condensation networks induced by strict Nash networks allow us to partition players into groups that obtain the same amount of resources from the network. This approach is appealing in the sense that it allows us to identify the equilibrium networks in a very succint manner. We illustrate our approach using a noncooperative model of network formation that we call the partner heterogeneity model.

Noncooperative models of network formation were introduced to the literature by Bala and Goyal [1]. Our paper focuses on a class of networks models called one-way flow models. In this set up if player i forms an arc with player j, then player i obtains resources from j but not vice versa. player i also obtains the resources of all the players that j observes directly or through a sequence of indirect connections in the directed network. The stability notion we use here is Nash equilibrium.[1] The authors show that if players ' payoffs are increasing in the number of other players accessed, and decreasing in the number of arcs formed, then a strict Nash network is either a wheel (a connected network in which each player creates and receives one link or arc), or the empty network. It follows that, in the strict equilibria, players are always in a symmetric position with regard to the resources they obtain. This conclusion is not in line with most of the empirical findings in the networks literature where networks typically have asymmetric architectures, and hence players do not obtain the same resources.

Recently several papers have examined models of networks formation with heterogeneity.[2] The paper that relates most closely to our work is by Galeotti [9]. He studies situations where players are heterogeneous with regard to the costs of linking and the values of accessing other players . More precisely, he introduces two kinds of heterogeneity.
In the first kind, each player i obtains (incurs) the same value (cost) from every player k. In the paper

[1] Jackson and Wolinsky [2] introduced another stability notion: *pairwise stability*. A network is pairwise stable if no couple of players has an incentive to form a link and no player has an incentive to remove a link she is involved in.

[2] Gilles and Johnson [4] consider link costs that are increasing in the spatial distance between players while McBride [5] focuses on value heterogeneity and partial information regarding the network structure. Haller and Sarangi [6] propose a model of heterogeneous link reliability and Hojman and Szeidl [7] develop a general model of decay where the resources obtained by a player depend on the distance between the players in the network. Galeotti et al. [?] introduce heterogeneity in the two-way flow connections model initiated by Bala and Goyal [1] by allowing costs and benefits of links to depend on the identity of the player who is forming the links. Finaly, Billand et al. [8] examines the impact of the partner heterogeneity on the size of the set of strict Nash networks in the two-way flow model.

we call this framework *the player heterogeneity framework*. Galeotti shows that if the parameters of the model are such that there is no isolated players in strict equilibrium, then there are two cases: either each player obtains the resources of *all* other players , or the set of players is partitioned into two groups. The members of the first group obtain the resources of the entire population while each member of the second group only obtains her own resources. These results leads to the two following conclusions. First, player heterogeneity can produce asymmetries between players with regard to the resources they obtain in a strict equilibrium. Second, player heterogeneity leads to very specific situations concerning these asymmetries: either there are no asymmetries between players since each agent obtain the resources of all others, or the asymmetries are "very strong" since a group of players obtains the resources of all the population and the other players do not benefit from the network formation.
The second kind of heterogeneity allows costs and values to depend on the identity of partner involved in the relationship.[3] Networks with partner heterogeneity property can be frequently encountered in the real world. For example, on the World Wide Web players can access the webpage of other players in order to obtain information without the consent of the webmaster. Similarly, peer to peer softwares like Kazaa or Emule generally allow to a player i to obtain resources from a player j without the explicit consent of the latter. Moreover, an important aspect of these situations is that the costs or benefits that a player obtains from another player i depend on the characteristics of i, that is benefits and costs of links are *partner dependent*. Thus, some homepages are easier to access on the web as compared to others. Likewise due to the speed connection of players,[4] some users are easier to access with peer-to-peer softwares as compared to others.

Galeotti (pg.173, [9]) writes *"An open question, which is left for further research, is whether we can say something systematic about the architectural properties of equilibrium networks in partner specific heterogeneous models."* Our goal here is twofold: (i) to provide an answer to this question, and (ii) to illustrate the usefulness of the condensation network studying situations with heterogeneity.

[3]Of course a third kind of heterogeneity allows costs and values to depend on the identity of both players involved in the relationship, that is the cost and value have two degrees of freedom. In this case, for any minimal network (networks where there is no superfluous links with regard to the resources obtained by players) there exist parameters for which it is a strict Nash network. Here all types of asymmetries can arise in strict Nash networks with regard to the resources obtained by players .

[4]Generally, download speed is higher than upload speed.

Consequently, in the paper we focus on the condensation networks induced by strict Nash networks. Relative to the notion of strict Nash networks which focus on individual players and flows of resources between them, the condensation networks induced by strict Nash networks are well designed to highlight the asymmetries in resources obtained by groups of players in networks. We then use properties of binary relations, specifically, the chain and inf-semi lattice to characterize these condensation networks. This allows us to rank groups of players with regard to the set of resources obtained. Hence, we are able to characterize resources asymmetries between groups of players in equilibrium. In the paper, we examine two different frameworks. In the first one, values are partner dependent while costs are homogeneous. In the second one, both values and costs are partner dependent.

Our main findings are as follows.

1. If value is partner dependent while the cost of forming links is homogeneous, then there are either 1 or n groups of players (with regard to the values obtained) in strict Nash networks. Moreover, when there are n groups, we can define a *chain* with regard to the obtained resources relation over the set of players . In other words for any two players i and j, either i obtains the resources of j, or j obtains the resources of i.

2. If both value and cost are partner dependent, then we obtain situations where there exists a specific partial order relation between groups of players with regard to the resources they obtain: a *inf-semi-lattice*. That is two groups of players always have a greatest lower bound with regard to the resources they obtain.

Our paper contributes to the literature in the following way. First, we show how condensation networks provide an alternative approach to characterize equilibrium networks with heterogeneity. Second, we establish that the dichotomy concerning the asymmetries in strict Nash networks obtained by Galeotti [9] results from the player heterogeneity assumption and does not hold anymore when we introduce partner heterogeneity. More precisely, it is not necessary to have two degrees of freedom in cost or in value to obtain intermediate ranges of asymmetries between players with regard to the resources that they obtain in strict Nash networks. To sum up, we know from Bala and Goyal [1] that heterogeneity is a necessary condition for strict Nash networks to exhibit asymmetries in resources obtained by players when there is no imperfection concerning the resources flows. We establish that the kind of asymmetries that we obtain depends not only on the number of degree of freedom of the value and cost parameters but also on the nature of this heterogeneity.

The rest of the paper is organized as follows. Section 2 introduces the model setup and the notions used in the paper. Section 3 presents the results under partner value heterogeneity with homogeneous cost. Section 4 deals with partner value heterogeneity with partner dependent cost. Section 5 concludes.

2 Model setup

Networks definitions. A network $\boldsymbol{g}$ is an ordered pair of disjoint sets (V, A) such that A is a subset of the set $V \times V$ of ordered pairs of V. The set V is the set of vertices and A is the set of arcs. Let $\mathcal{G}$ be the set of directed networks. If $\boldsymbol{g}$ is a directed network, then $V = V(\boldsymbol{g})$ is the vertex set of $\boldsymbol{g}$, and $A = A(\boldsymbol{g})$ is the arc set. An ordered pair $(i, j) \in A(\boldsymbol{g})$ is said to be an arc directed from i to j and is denoted $i\ j$. If there is an arc from $i \in V(\boldsymbol{g})$ to $j \in V(\boldsymbol{g})$ in $\boldsymbol{g}$, then i is a predecessor of j in $\boldsymbol{g}$. Let $A_i(\boldsymbol{g}) \subset A(\boldsymbol{g})$, $A_i(\boldsymbol{g}) = \{i\ j \mid j \in N\}$ be the set of arcs directed from i to another vertex. For a directed network, $\boldsymbol{g}$, a *path* $P_{i,j}(\boldsymbol{g})$ in $\boldsymbol{g}$ from (the initial) vertex i to (the terminal) vertex j is an alternating sequence of vertices and arcs: $i_0, i_0\ i_1, i_1, i_1\ i_2, \ldots, i_{\ell-1}\ i_\ell, i_\ell$ such that $i_0 = i$, $i_\ell = j$. A *cycle* is obtained from a path on adding an arc from the terminal vertex to the initial vertex.
We say that $\boldsymbol{g}'$ is a *sub-network* of $\boldsymbol{g}$ if $V(\boldsymbol{g}') \subset V(\boldsymbol{g})$ and $\boldsymbol{g}'$ contains all arcs of $\boldsymbol{g}$ that join two vertices in $V(\boldsymbol{g}')$. A network is *connected* if for every pair (i, j) of distinct vertices there is a path from i to j. A *maximal connected* sub-network of $\boldsymbol{g}$ is a *component* of $\boldsymbol{g}$. The *empty network*, $\boldsymbol{g}^e$, is a network which contains no link between distinct vertices. The network $\boldsymbol{g}$ is a *tail star* if the set of vertices can be partitioned into two groups $N_1 = \{1, \ldots, k\}$, $N_2 = \{k+1, \ldots, n\}$ such that for all vertices $i \in N_1$, we have $i+1\ i \in A(\boldsymbol{g})$, and $i\ j \notin A(\boldsymbol{g})$ otherwise, and for all vertices $i \in N_2$, we have $i\ k \in A(\boldsymbol{g})$, and $i\ j \notin A(\boldsymbol{g})$ for all $j \neq k$. A *line network* $\boldsymbol{g}$ is a *tail star* where $N_2 = \emptyset$. A *center sponsored star* is a network where there is a link from a vertex i_0 to all vertices $j \neq i_0$ and there are no other links. Two networks $\boldsymbol{g}$ and $\boldsymbol{g}'$ are *isomorphic* if they have the same number p of vertices and if we can order their vertices respectively $i_1, i_2, \ldots, i_p$ and $j_1, j_2, \ldots, j_p$ so that for any k and ℓ, arc $i_k\ i_\ell$ is in $A(\boldsymbol{g})$ iff arc $j_k\ j_\ell$ is in $A(\boldsymbol{g}')$.

We now present the notion of condensation network given by Harary, Norman and Cartwright (HNC, [10]). Let $F : 2^{V(\boldsymbol{g})} \to \{1, \ldots, 2^n\}$ be a one to one mapping. F maps any subset of the set of vertices of $\boldsymbol{g}$ to a number.

The network $\boldsymbol{g}^\star$ is a **condensation network** induced by the directed network $\boldsymbol{g}$ if the set of vertices satisfies Property 1 and the set of arcs satisfies Property 2.

Property 1. Let $X \subseteq V(\boldsymbol{g})$. $F(X) \in V(\boldsymbol{g}^\star)$ iff (i) there is a path from any $i \in X$ to any $j \in X$ in $\boldsymbol{g}$ and (ii) there is no Y, $Y \supset X$ such that there is a path from some $i \in Y$ to some $j \in Y$ in $\boldsymbol{g}$.

Property 2. Let $F(X) \in V(\boldsymbol{g}^\star)$ and $F(Y) \in V(\boldsymbol{g}^\star)$. There is an arc from $F(X)$ to $F(Y)$ iff there exist vertices $i \in X$ and $j \in Y$ such that there is an arc from i to j in $A(\boldsymbol{g})$.

Clearly, these properties imply that the set of vertices of the condensation network $\boldsymbol{g}^\star$ is constructed by using the components of $\boldsymbol{g}$ to partition the set of vertices V($\boldsymbol{g}$). Moreover, we know from HNC ([10], Theorem 3.2, pg.55) that every vertex is contained in exactly one component and each arc is contained in at most one component. It follows that it is always possible to construct a condensation network from any network.

We now give an example of a condensation network $\boldsymbol{g}^\star$ induced by a network $\boldsymbol{g}$. Let $N = \{1, \ldots, 7\}$ be the set of vertices of $\boldsymbol{g}$ and the arcs of $\boldsymbol{g}$ be drawn in Figure 1. Then the condensation network is drawn in Figure 1. We observe that vertices 1, 2, 4, and 6 are in the same component in $\boldsymbol{g}$, consequently they are associated with the same vertex in $\boldsymbol{g}^\star$. We assume that $F(\{1, 2, 4, 6\}) = 9$. Similarly, vertices 5 and 7 are in the same component in $\boldsymbol{g}$, consequently they are associated with the same vertex in $\boldsymbol{g}^\star$. We set $F(\{5, 7\}) = 10$. The isolated vertex 3 is associated with $F(\{3\}) = 8$ in $\boldsymbol{g}^\star$. Finally, Property 2 concerning the links between vertices in the condensation networks implies that there is an arc from 10 to 8, from 10 to 9 and from 9 to 8 in $\boldsymbol{g}^\star$.

The example below shows that architectures of condensations networks can be simpler than the

Network $\boldsymbol{g}$ Condensation network $\boldsymbol{g}^\star$ induced by $\boldsymbol{g}$

Figure 1: Condensation network

architectures of the initial network.

Players and strategies. Since we use the notion of condensation networks, to avoid confusion, we will make a distinction between the set of players (or decision makers) and the set of vertices. In particular, we will not assume that the set of players and the set of vertices are necessarily one and the same. Let $N = \{1, \dots, n\}$ be the set of players. The original network $\boldsymbol{g}$, that is the network formed by the players, is such that $V(\boldsymbol{g}) = N$. In $\boldsymbol{g}$ the relations among the players are formally represented by the arcs of $\boldsymbol{g}$. Let $G_i = \{i\ j \mid j \in N \setminus \{i\}\}$ be the set of arcs that player i can form with other players. In our context, each player $i \in N$ chooses a strategy which consists in forming arcs: $A_i(\boldsymbol{g}) \in 2^{G_i}$. If $i\ j \in A_i(\boldsymbol{g})$, then player i has formed an arc with player j in $\boldsymbol{g}$, otherwise player i has not formed an arc with player j. We only focus on pure strategies in this paper. Notice that the set of arcs between distinct players of the network $\boldsymbol{g}$ is $A(\boldsymbol{g}) = \bigcup_{i \in N} A_i(\boldsymbol{g})$. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $A_{-i}(\boldsymbol{g}) = \bigcup_{j \in N \setminus \{i\}} A_j(\boldsymbol{g})$ denote the strategy employed by all players except i. The set of arcs between distinct players of network $\boldsymbol{g}$ can be also written as $A(\boldsymbol{g}) = A_i(\boldsymbol{g}) \bigcup A_{-i}(\boldsymbol{g})$. Hence if $A_i(\boldsymbol{g}) = \emptyset$, then $A_{-i}(\boldsymbol{g}) = A(\boldsymbol{g})$. It follows that $A_{-i}(\boldsymbol{g})$ is also the set of arcs between distinct players of the network obtained from $\boldsymbol{g}$ when all the arcs formed by i are removed. To simplify notation, we let $A_i(\boldsymbol{g}) \cup \{i\ j\} = A_i(\boldsymbol{g}) + i\ j$ and $A_i(\boldsymbol{g}) \setminus \{i\ j\} = A_i(\boldsymbol{g}) - i\ j$. For consistency, we let $A(\boldsymbol{g}) \cup \{i\ j\} = A(\boldsymbol{g}) + i\ j$ and $A(\boldsymbol{g}) \setminus \{i\ j\} = A(\boldsymbol{g}) - i\ j$.

Payoffs. To complete our strategic form game of network formation, we now specify the payoffs. The amount of resources that each player obtains in a network $\boldsymbol{g}$ depends on the architecture of $\boldsymbol{g}$. More precisely, an arc $i\ j$ that player i forms with player j allows player i to get resources from player j. However, since we are in the one-way flow or directed network model, this link does not allow j to obtain resources from i. Also, player i receives information from other players not only through direct arcs, but also via indirect arcs. To be precise, information flows from player j to player i if i and j are linked by a path from i to j in $\boldsymbol{g}$. Let $N_i(A(\boldsymbol{g})) = \{j \in N \mid \text{there is a path from } i \text{ to } j \text{ in the network } (N, A(\boldsymbol{g}))\}$ be the set of players from whom i accesses resources. By construction, we have for each i, $i \in N_i(A(\boldsymbol{g}))$. Since our goal is to analyze the partner heterogeneity model each player j allows each player i to obtain the same resources from j, $v_j > 0$, when $j \in N_i(\boldsymbol{g})$. The formation of an arc with player j also implies a cost c_j for all players $i \in N \setminus \{j\}$. Each player i always obtain her own resources. However we wish to focus only on the impact of the network on the payoff of players. Hence we do not take into account the resources that each player i obtains from herself in her payoff function. Indeed, a player can obtain her own resources even if she forms no arcs, *i.e.* there is no network. Finally, to facilitate the comparison between the partner heterogeneity framework and the player heterogeneity framework of Galeotti [9], we use the following linear payoff function:[5]

$$\pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) = \sum_{j \in N_i(A(\boldsymbol{g}))\setminus\{i\}} v_j - \sum_{j: ij \in A_i(\boldsymbol{g})} c_j. \tag{1}$$

Nash networks and strict Nash networks. The strategy $A_i(\boldsymbol{g})$ is said to be a best response of player i against the strategy $A_{-i}(\boldsymbol{g})$ if:

$$\pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) \geq \pi_i(E_i, A_{-i}(\boldsymbol{g})) \text{ for all } E_i \in 2^{G_i}. \tag{2}$$

The set of all of player i's best responses to $A_{-i}(\boldsymbol{g})$ is denoted by $\mathcal{BR}_i(A_{-i}(\boldsymbol{g}))$. A network $\boldsymbol{g}$ is said to be a *Nash network* if $A_i(\boldsymbol{g}) \in \mathcal{BR}_i(A_{-i}(\boldsymbol{g}))$ for each player $i \in N$. We define a strict best response and a strict Nash network by replacing '$\geq$' by '$>$'.

It is obvious that in a strict Nash network $\boldsymbol{g}$, player i does not form an arc with player j if she obtains the resources of j in the network $\boldsymbol{g}'$ with $A(\boldsymbol{g}') = A(\boldsymbol{g}) - i\ j$. We call this property the *basic property of strict Nash networks* (BPSN). The BPSN implies that there is at most one link between two vertices of a condensation network induced by a strict Nash network.

Relations: notation and definitions. We define $\succeq_{\boldsymbol{g}}$ as the following binary relation on $V(\boldsymbol{g})$: $x \succeq_{\boldsymbol{g}} y$ iff there is a path from vertex x to vertex y in $\boldsymbol{g}$. We suppose that there always exists a path from a vertex to itself in $\boldsymbol{g}$. In particular, this definition implies that in the original network $\boldsymbol{g}$, where $V(\boldsymbol{g}) = N$, we have $i \succeq_{\boldsymbol{g}} j$ iff $j \in N_i(\boldsymbol{g})$. In that case $i \succeq_{\boldsymbol{g}} j$ means that player i obtains resources from player j, conversely the dual relation $i \not\succeq_{\boldsymbol{g}} j$ means that player i does not obtain resources from player j. In the rest of the paper, the main results concern the properties of $\succeq_{\boldsymbol{g}}$ over N in the original network $\boldsymbol{g}$ and the properties of $\succeq_{\boldsymbol{g}^\star}$ over $V(\boldsymbol{g}^\star)$ in the condensation network $\boldsymbol{g}^\star$. Let us recall some important classes of binary relations.

[5]The results hold qualitatively on relaxing this linearity assumption. However, the payoff function must satisfy the following property: If $\pi_i(A_i(\boldsymbol{g}) + i\ k, A_{-i}(\boldsymbol{g})) - \pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) > 0$, and $N_j(\boldsymbol{g}) \cap N_k(\boldsymbol{g}) = \emptyset$, then $\pi_j(A_j(\boldsymbol{g}) + j\ k, A_{-j}(\boldsymbol{g})) - \pi_j(A_j(\boldsymbol{g}), A_{-j}(\boldsymbol{g})) > 0$. In other words, if player i has an incentive to form an arc with player k in $\boldsymbol{g}$ and player j obtains no resources from k in $\boldsymbol{g}$, then player j must also have an incentive to form an arc with k in $\boldsymbol{g}$.

The relation $\succeq_{\boldsymbol{g}}$ is reflexive over $V(\boldsymbol{g})$ if $x \succeq_{\boldsymbol{g}} x$ for all $x \in V(\boldsymbol{g})$. The relation $\succeq_{\boldsymbol{g}}$ is total if $x \succeq_{\boldsymbol{g}} y$ or $y \succeq_{\boldsymbol{g}} x$ for all $x, y \in V(\boldsymbol{g})$. It is symmetric over $V(\boldsymbol{g})$ if $x \succeq_{\boldsymbol{g}} y$ implies $y \succeq_{\boldsymbol{g}} x$ for all $x, y \in V(\boldsymbol{g})$. It is antisymmetric over $V(\boldsymbol{g})$ if $x \succeq_{\boldsymbol{g}} y$ and $y \succeq_{\boldsymbol{g}} x$ imply $x = y$ for all $x, y \in V(\boldsymbol{g})$. The relation $\succeq_{\boldsymbol{g}}$ over $V(\boldsymbol{g})$ is transitive if $x \succeq_{\boldsymbol{g}} y$ and $y \succeq_{\boldsymbol{g}} z$ imply $x \succeq_{\boldsymbol{g}} z$ for all x, y and z in $V(\boldsymbol{g})$. It is obvious, by the construction of $N_i(\boldsymbol{g})$, that $\succeq_{\boldsymbol{g}}$ is reflexive and transitive over N in the original network $\boldsymbol{g}$. A relation that is reflexive, symmetric, and transitive is called an equivalence relation. An equivalence relation specifies how to partition a set into subsets called equivalence classes. A *partially ordered set* $(V(\boldsymbol{g}), \succeq_{\boldsymbol{g}})$ is a set $V(\boldsymbol{g})$ on which there is a relation $\succeq_{\boldsymbol{g}}$ that is reflexive, antisymmetric, and transitive. A partially ordered set $(V(\boldsymbol{g}), \succeq_{\boldsymbol{g}})$ is a *chain* if $\succeq_{\boldsymbol{g}}$ is total. A *Hasse diagram* is a graphical representation of a partially ordered set.

Suppose that $(V(\boldsymbol{g}), \succeq_{\boldsymbol{g}})$ is a partially ordered set and X is a subset of $V(\boldsymbol{g})$. If x is in $V(\boldsymbol{g})$ and $y \succeq_{\boldsymbol{g}} x$ for each y in X, then x is a lower bound for X. If the set of lower bounds of X has a greatest element, then this greatest lower bound of X is the infimum of X. If two elements x and y, of a partially ordered set $V(\boldsymbol{g})$, have a greatest lower bound, it is their meet and it is denoted $x \wedge y$. A partially ordered set $V(\boldsymbol{g})$ that contains the meet of each pair of its elements is a *inf-semi-lattice*.[6]

It is worth noting that $\boldsymbol{g}^\star$, the condensation of a network $\boldsymbol{g}$, allows us to capture the set of players who obtain the same resources. More precisely, all players who belong to X with $F(X) \in V(\boldsymbol{g}^\star)$ obtain the same resources. In other words, each vertex of $\boldsymbol{g}^\star$ is an equivalence class of $\succeq_{\boldsymbol{g}}$ over N.

Finally, we say that vertex $x \in V(\boldsymbol{g})$ is a source in $\boldsymbol{g}$ if for all $y \in V(\boldsymbol{g}) \setminus \{x\}$ we have $y \succeq_{\boldsymbol{g}} x$. It is worth noting that a source of an acyclic network $\boldsymbol{g}$ does not form any arc.[7]

3 Partner heterogeneous values and homogeneous costs

In this section we will order the players using the sets of resources they obtain in equilibrium. Therefore we focus only on non-empty strict Nash networks (and their condensation networks). We begin with a property which is satisfied by all strict Nash networks in situations where values are partner dependent and costs are homogeneous, that is $c_j = c$ for all players $j \in N$.

[6]See Birkhoff [11] for additional information on semi-lattices. In Birkhoff inf-semi-lattices are called meet-semilattices.

[7]This result follows Theorem 4.3, pg. 89, in Harary, Norman and Cartwright [10].

Proposition 1 *Suppose the payoff function of each player i satisfies (1), with $c_j = c$ for all $j \in N$. If $\boldsymbol{g}$ is a non-empty strict Nash network, then $|V(\boldsymbol{g}^\star)| \in \{1, n\}$.*

Proof Let $\boldsymbol{g}$ be a non-empty strict Nash network. To introduce a contradiction suppose $|V(\boldsymbol{g}^\star)| \in \{2, \ldots, n-1\}$. Given that $|V(\boldsymbol{g})| = n$ there are two possibilities to obtain such a result . Either there exist $F(X)$ and $F(Y)$ in $V(\boldsymbol{g}^\star)$ such that $X, Y \in 2^N$ and both $|X|, |Y| > 1$, or there exists $F(X)$ in $V(\boldsymbol{g}^\star)$ such that $X \in 2^N$ and $n > |X| > 1$, and for all $F(Y)$ in $V(\boldsymbol{g}^\star) \setminus \{F(X)\}$, we have $|Y| = 1$. We deal successively with these two possibilities.

1. Suppose $F(X)$ and $F(Y)$ in $V(\boldsymbol{g}^\star)$ with $|X|, |Y| > 1$. There are two cases: either (i) there is a path from $F(X)$ (or $F(Y)$) to $F(Y)$ (respectively $F(X)$) in $\boldsymbol{g}^\star$ or (ii) such a path does not exist.

   (i) Suppose wlog that there is a path from $F(X)$ to $F(Y)$ with $i \in X$ and $j \in Y$, that is $j \in N_i(\boldsymbol{g})$. It follows that there is a path from k to j, with $k \in N \setminus Y$. Consequently, there are players j' and k' in N such that $k'\ j' \in A(\boldsymbol{g})$ with $k' \in N \setminus Y$ and $j' \in Y$. Moreover, since $j' \in Y$, $F(Y) \in V(\boldsymbol{g}^\star)$ and $|Y| > 1$, there is $j'' \in Y$ such that $j''\ j' \in A(\boldsymbol{g})$. Let $\boldsymbol{g}'$ be the network where $A(\boldsymbol{g}') = A(\boldsymbol{g}) + j''\ i - j''\ j'$. In $\boldsymbol{g}'$, j'' incurs the same cost as in $\boldsymbol{g}$ since she forms the same number of arcs, and obtains more resources since by construction $N_i(\boldsymbol{g}') \supset N_{j'}(\boldsymbol{g})$. It follows that $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

   (ii) Now suppose that there is no path between $F(X)$ and $F(Y)$. Since $F(X) \in V(\boldsymbol{g}^\star)$ (respectively $F(Y) \in V(\boldsymbol{g}^\star)$) and $|X| > 1$ (respectively $|Y| > 1$), there are $i, i' \in X$ such that $i\ i' \in A(\boldsymbol{g})$ (respectively $j, j' \in Y$ such that $j\ j' \in A(\boldsymbol{g})$). We show that $\boldsymbol{g}$ is not a strict Nash network. To introduce a contradiction suppose $\boldsymbol{g}$ is a strict Nash network. Assume player i chooses strategy $E_i = A_i(\boldsymbol{g}) - i\ i' + i\ j'$. By using the fact that players j and j' belong to Y, with $F(Y) \in V(\boldsymbol{g}^\star)$, it is clear that the marginal payoff obtained by player i is:

$$\pi_i(E_i, A_{-i}(\boldsymbol{g})) - \pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) = \sum_{\ell \in N_j(A(\boldsymbol{g}))} v_\ell - \sum_{\substack{\ell \in N_i(A_{-i}(\boldsymbol{g}) + i\ i'), \\ \ell \notin N_i(A(\boldsymbol{g}) - i\ i')}} v_\ell \tag{3}$$

   which is strictly negative since $\boldsymbol{g}$ is a strict Nash network. Suppose player j chooses strategy $E_j = A_j(\boldsymbol{g}) - j\ j' + j\ i$. The marginal payoff obtained by player j is:

$$\pi_j(E_j, A_{-j}(\boldsymbol{g})) - \pi_j(A_j(\boldsymbol{g}), A_{-j}(\boldsymbol{g})) = \sum_{\ell \in N_i(A(\boldsymbol{g}))} v_\ell - \sum_{\substack{\ell \in N_j(A_{-j}(\boldsymbol{g}) + j\ j'), \\ \ell \notin N_j(A(\boldsymbol{g}) - j\ j')}} v_\ell . \tag{4}$$

which is strictly negative since $\boldsymbol{g}$ is a strict Nash network. Then, we obtain a contradiction since by summing equations (3) and (4) we obtain:

$$\sum_{\ell \in N_j(A(\boldsymbol{g}))} v_\ell - \sum_{\substack{\ell \in N_i(A_{-i}(\boldsymbol{g})+i\ i'),\\ \ell \notin N_i(A(\boldsymbol{g})-i\ i')}} v_\ell + \sum_{\ell \in N_i(A(\boldsymbol{g}))} v_\ell - \sum_{\substack{\ell \in N_j(A_{-j}(\boldsymbol{g})+j\ j'),\\ \ell \notin N_j(A(\boldsymbol{g})-j\ j')}} v_\ell > 0.$$

2. Next suppose $F(X) \in V(\boldsymbol{g}^\star)$, with $n > |X| > 1$ and let $F(Y) \in V(\boldsymbol{g}^\star)$, with $|Y| = 1$. There are three cases.
   (i) There is a path from $F(Y)$ to $F(X)$ in $\boldsymbol{g}^\star$. In such a case, there are players $i, i' \in X$ and $j \in N \setminus X$ such that $i\ i'$ and $j\ i$ belong to $A(\boldsymbol{g})$. By partner value heterogeneity and cost homogeneity player $j \in N \setminus X$ obtains the same payoff if she replaces her link with i by an arc with i'. Therefore, $\boldsymbol{g}$ is not a strict Nash network.
   (ii) There is a path from $F(X)$ to $F(Y)$ in $\boldsymbol{g}^\star$. Since $|X| > 1$ and $F(X) \in V(\boldsymbol{g}^\star)$ there are players i and i' in X such that $i\ i' \in A(\boldsymbol{g})$. Since there is a path from $F(X)$ to $F(Y)$, either player i or player i' has formed an arc with a player $j \notin X$ in $\boldsymbol{g}$. Suppose without loss of generality that player i has formed an arc with player j in $\boldsymbol{g}$. Then there is no path, in the network associated with $A(\boldsymbol{g}) - i\ j$, from i' to j otherwise BSNP is violated. It follows that $\pi_j(A_j(\boldsymbol{g}) + j\ i', A_{-j}(\boldsymbol{g})) - \pi_j(A_j(\boldsymbol{g}), A_{-j}(\boldsymbol{g})) \geq \pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) - \pi_i(A_i(\boldsymbol{g}) - i\ i', A_{-i}(\boldsymbol{g})) > 0$, that is player j obtains the same resources from player i' as player i and incurs the same cost when she forms the arc $j\ i'$. Consequently, $\boldsymbol{g}$ is not a strict Nash network, a contradiction.
   (iii) There is no path between $F(X)$ and $F(Y)$ in $\boldsymbol{g}^\star$. Then we can use the same argument as in the point 1(ii) above.

□

We now recall two results of directed graph theory useful for Proposition 2.

Theorem 1 *(HNC, Theorem 3.7, pg.63, [10]) The following statements are equivalent for any directed network $\boldsymbol{g}$.*

1. *$\boldsymbol{g}$ is acyclic, that is, has no cycles.*
2. *$\boldsymbol{g}$ and $\boldsymbol{g}^\star$ have the same number of vertices.*
3. *$\boldsymbol{g}$ and $\boldsymbol{g}^\star$ are isomorphic.*

Theorem 2 *(HNC, Theorem 3.9, pg.65, [10]) The following statements are equivalent for any directed network $\boldsymbol{g}$.*

1. *There is a path from i to j, for any players i and j in $\boldsymbol{g}$.*

2. *$\boldsymbol{g}^\star$ consists of exactly one vertex.*

In the following proposition, we will not focus on the condensation network but on the original network $\boldsymbol{g}$. Note that by Theorem 1 in the non-trivial case (that is $|V(\boldsymbol{g}^\star)| = n$) the original network $\boldsymbol{g}$ and its condensation network $\boldsymbol{g}^\star$ are isomorphic.

Proposition 2 *Suppose the payoff function of each player i satisfies (1) with $c_j = c$ for all $j \in N$. Let $\boldsymbol{g}$ be a non-empty strict Nash network. If $|V(\boldsymbol{g}^\star)| = 1$, then $\succeq_{\boldsymbol{g}}$ is an equivalence relation over N. If $|V(\boldsymbol{g}^\star)| = n$, then $(N, \succeq_{\boldsymbol{g}})$ is a chain.*

Proof Let $\boldsymbol{g}$ be a non empty strict Nash network. Recall that, by construction, $\succeq_{\boldsymbol{g}}$ is reflexive and transitive over N.

1. Suppose $|V(\boldsymbol{g}^\star)| = 1$, that is $\boldsymbol{g}^\star$ consists of exactly one vertex. We show that $\succeq_{\boldsymbol{g}}$ is symmetric. By Theorem 2 there is a path from i to j and a path from j to i, for any players i and j in $\boldsymbol{g}$. It follows that for all $i, j \in N$, we have $i \succeq_{\boldsymbol{g}} j$ and $j \succeq_{\boldsymbol{g}} i$.

2. Suppose $|V(\boldsymbol{g}^\star)| = n$, we need to show that $\succeq_{\boldsymbol{g}}$ is antisymmetric and total over N. Clearly, $\boldsymbol{g}^\star$ and $\boldsymbol{g}$ have the same number of vertices: n. By Theorem 1, $\boldsymbol{g}$ is acyclic. Consequently, if $i \neq j$, then $i \not\succeq_{\boldsymbol{g}} j$ or $j \not\succeq_{\boldsymbol{g}} i$, that is $\succeq_{\boldsymbol{g}}$ is antisymmetric over N. We now show that $\succeq_{\boldsymbol{g}}$ is total over N. In other words, we show that we have either $N_i(A(\boldsymbol{g})) \subseteq N_j(A(\boldsymbol{g}))$, or $N_i(A(\boldsymbol{g})) \supseteq N_j(A(\boldsymbol{g}))$ for any players $i, j \in N$.
   First, since $\boldsymbol{g}$ (i) is a non empty strict Nash network, and (ii) is acyclic, there are players $i, j \in N$ such that $i\ j \in A(\boldsymbol{g})$ and $N_j(A(\boldsymbol{g})) = \{j\}$. If player i has an incentive to form an arc with player j in $\boldsymbol{g}$, then each player k such that $j \notin N_k(A(\boldsymbol{g}))$ has an incentive to form an arc with player j in $\boldsymbol{g}$. Indeed, $\pi_k(A_k(\boldsymbol{g}) + k\ j, A_{-k}(\boldsymbol{g}) - \pi_j(A_k(\boldsymbol{g}), A_{-k}(\boldsymbol{g}) = \pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) - \pi_i(A_i(\boldsymbol{g}) - i\ j, A_{-i}(\boldsymbol{g})) > 0$ by partner heterogeneity of value and homogeneity of cost. It follows that $N_i(A(\boldsymbol{g})) \cap N_j(A(\boldsymbol{g})) \neq \emptyset$ for each $i, j \in N$.
   Second, we show that there is no player i such that both players j and k have formed an arc with her, that is each player i has at most one predecessor in $\boldsymbol{g}$. To introduce a contradiction

suppose that there is a player i with two predecessors j and k. Then, player j has an incentive to form an arc with k instead of i since $v_k > 0$ for all $k \in N$. Since there is a player j such that $j \in N_i(A(\boldsymbol{g}))$ for all $i \in N$ and each player i has at most one predecessor, we obtain that $\succeq_{\boldsymbol{g}}$ is total over N by using induction.

□

We now show through the following example that there exist some parameters such that $(N, \succeq_{\boldsymbol{g}})$ is a chain and $\boldsymbol{g}$ is a strict Nash network.

Example 1 Let $N = \{1, 2, 3\}$, $v_1 = 4$, $v_2 = v_3 = 1$, and $c = 3$. Then the network $\boldsymbol{g}$ such that $A(\boldsymbol{g}) = \{2\ \ 1, 3\ \ 2\}$ is strict Nash.

Proposition 2 showed that two types of situations arise in non-empty strict Nash networks. In the first one, $|V(\boldsymbol{g}^\star)| = 1$, all players belong to the same equivalence class. In that case, each player obtains the resources of all other players.
In the second one, $|V(\boldsymbol{g}^\star)| = n$, $(N, \succeq_{\boldsymbol{g}})$ is a chain, that is for each player i and each player j, either player i obtains the resources of player j (and player j does not obtain resources of i) or player j obtains the resources of player i (and player i does not obtain resources of j). It follows that if costs are homogeneous and values heterogeneous, then players are either perfectly symmetric or asymmetric. In the symmetric case, they obtain the same resources. When they are asymmetric however, we have a ranking of players where the difference between the players resources changes incrementally giving us a "smooth" asymmetry. Notice that in partner heterogeneity models owning most valuable resources is detrimental to the owner, say player j, in the network. Also observe that in overall terms, however, player j does have the highest resources.

We can compare our result with the result obtained by Galeotti in the player heterogeneity framework. Galeotti provides results for cost and value player heterogeneity (Proposition 3.1, pg.169, [9]). These results are preserved when the cost is homogeneous. Clearly, in Galeotti (Proposition 3.1, [9]) the condensation networks induced by non-empty strict Nash networks have $x \in \{1, \ldots, n\}$ vertices. By contrast, in the partner heterogeneity framework, the condensation networks induced by non-empty strict Nash networks have either 1 or n vertices. Moreover, in Galeotti, there are some situations where no player obtains the resources of all other players, that is $(N, \succeq_{\boldsymbol{g}})$ has no maximal element. Indeed, in the Galeotti framework, some players can be isolated. Hence, it is not possible to compare some players with others with regard to the set of resources they obtain. In other words, condensation networks induced by non-empty strict Nash networks contain groups of players who do not share resources. This result differs from our result since we find that for all players i and j, either player i obtains resources of j or j obtains resources of i.

We now provide the architectures of condensation networks induced by non-empty strict Nash networks. Obviously, if $|V(\boldsymbol{g}^\star)| = 1$, then the condensation network is empty. The following corollary provides the architectures of condensation networks induced by non-empty strict Nash networks when $|V(\boldsymbol{g}^\star)| = n$.

Corollary 1 *Suppose the payoff function of each player i satisfies (1) with $c_j = c$ for all $j \in N$. If $|V(\boldsymbol{g}^\star)| = n$, then $\boldsymbol{g}^\star$ is a line network.*

Proof By Proposition 2, we know that if $|V(\boldsymbol{g}^\star)| = n$, then $\succeq_{\boldsymbol{g}}$ is a chain over N. It follows that $\boldsymbol{g}$ is a line network as the Hasse diagram of a chain. By Theorem 2, $\boldsymbol{g}$ and $\boldsymbol{g}^\star$ are isomorphic. Consequently, $\boldsymbol{g}^\star$ is a line network. $\square$

4 Partner heterogeneous values and costs

We now analyze situations where both costs and values are partner heterogeneous. In this setting, the architectures of strict Nash networks are much more complicated. Hence, the use of condensation networks induced by strict Nash networks will allow us to simplify the analysis of resources flows between players in equilibrium.

Proposition 3 *Suppose the payoff function of each player i satisfies (1) and let $\boldsymbol{g}$ be a strict Nash network. If $|X| > 1$, $F(X) \in V(\boldsymbol{g}^\star)$ and $F(Y) \in V(\boldsymbol{g}^\star)$, then $|Y| = 1$, for all $Y \neq X$.*

Proof Let $\boldsymbol{g}$ be a strict Nash network. To introduce a contradiction, suppose there are $|X| > 1$ and $|Y| > 1$, such that $F(X) \in V(\boldsymbol{g}^\star)$ and $F(Y) \in V(\boldsymbol{g}^\star)$. There are two cases: There is a path from $F(X)$ to $F(Y)$ (or a path from $F(Y)$ to $F(X)$) in $\boldsymbol{g}^\star$, or there is no path between $F(Y)$ to $F(X)$ in $\boldsymbol{g}^\star$. We examine these two cases successively.

1. Suppose wlog that there is a path from $F(X)$ to $F(Y)$ in $\boldsymbol{g}^\star$. Let $i \in X$ and $j \in Y$. We have $j \in N_i(\boldsymbol{g})$. It follows that there exists a player $k \in N \setminus Y$, such that there is a path from k

to j. Consequently, there are players j' and k' in $\boldsymbol{g}$ such that $k'\ j' \in A(\boldsymbol{g})$ with $k' \in N \setminus Y$ and $j' \in Y$. Moreover, since $j' \in Y$, $F(Y) \in V(\boldsymbol{g}^\star)$, and $|Y| > 1$, there is $j_0 \in Y$ such that $j_0\ j' \in A(\boldsymbol{g})$. Likewise, since $F(X) \in V(\boldsymbol{g}^\star)$, and $|X| > 1$, there exist players i and i' in X such that $i\ i' \in A(\boldsymbol{g})$. Let $\boldsymbol{g}'$ be the network where $A(\boldsymbol{g}') = A(\boldsymbol{g}) + j_0\ \ i' - j_0\ \ j'$. We have:

$$\pi_{j_0}(A_{j_0}(\boldsymbol{g}'), A_{-j_0}(\boldsymbol{g}')) - \pi_{j_0}(A_{j_0}(\boldsymbol{g}), A_{-j_0}(\boldsymbol{g})) = c_{j'} + \sum_{\substack{\ell \in N_i(A(\boldsymbol{g})),\\ \ell \notin N_{j'}(A(\boldsymbol{g}))}} v_\ell - c_{i'}$$

which is strictly negative since $\boldsymbol{g}$ is strict Nash. Let $\boldsymbol{g}''$ be the network where $A(\boldsymbol{g}'') = A(\boldsymbol{g}) + i\ \ j' - i\ \ i'$. We have:

$$\pi_i(A_i(\boldsymbol{g}''), A_i(\boldsymbol{g}'')) - \pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) = c_{i'} - \sum_{\substack{\ell \in N_i(A(\boldsymbol{g})),\\ \ell \notin N_i(A(\boldsymbol{g}''))}} v_\ell - c_{j'}$$

which is strictly negative since $\boldsymbol{g}$ is strict Nash. These conditions cannot be simultaneously satisfied since $N_{j'}(A(\boldsymbol{g})) \subseteq N_i(A(\boldsymbol{g}''))$. It follows that $\boldsymbol{g}$ is not a strict Nash network.

2. Second, suppose that there is no path between $F(X)$ and $F(Y)$ in $\boldsymbol{g}^\star$. Since $|X|, |Y| > 1$ and $F(X)$, $F(Y) \in V(\boldsymbol{g}^\star)$, there are players i, $i' \in X$ and j, $j' \in Y$ such that $i\ i' \in A(\boldsymbol{g})$ and $j\ j' \in A(\boldsymbol{g})$. Suppose player i replaces the link $i\ \ i'$ by the link $i\ \ j'$. By using the fact that players j and j' are in the same component, the marginal payoff obtained by player i is:

$$\sum_{\ell \in N_j(A(\boldsymbol{g}))} v_\ell - \sum_{\substack{\ell \in N_i(A_{-i}(\boldsymbol{g}) + i\ \ i'),\\ \ell \notin N_i(A(\boldsymbol{g}) - i\ \ i')}} v_\ell - c_{j'} + c_{i'} \tag{5}$$

which is strictly negative since $\boldsymbol{g}$ is strict Nash network.
Suppose player j replaces the link $j\ \ j'$ by the link $j\ \ i'$. The marginal payoff obtained by player j is:

$$\sum_{\ell \in N_i(A(\boldsymbol{g}))} v_\ell - \sum_{\substack{\ell \in N_j(A_{-j}(\boldsymbol{g}) + j\ \ j'),\\ \ell \notin N_j(A(\boldsymbol{g}) - j\ \ j')}} v_\ell - c_{i'} + c_{j'}. \tag{6}$$

which is strictly negative since $\boldsymbol{g}$ is strict Nash network.
We obtain a contradiction since by summing Equations 5 and 6, we have:

$$\sum_{\ell \in N_j(A(\boldsymbol{g}))} v_\ell - \sum_{\substack{\ell \in N_j(A_{-j}(\boldsymbol{g}) + j\ \ j'),\\ \ell \notin N_j(A(\boldsymbol{g}) - j\ \ j')}} v_\ell + \sum_{\ell \in N_i(\boldsymbol{g})} v_\ell - \sum_{\substack{\ell \in N_i(A_{-i}(\boldsymbol{g}) + i\ \ i'),\\ \ell \notin N_i(A(\boldsymbol{g}) - i\ \ i')}} v_\ell > 0.$$

$\square$

Proposition 3 highlights the fact that there is at most one equivalence class with several players with regard to the set of resources that players obtain.
Let us provide a useful result of graph theory for Proposition 4.

Theorem 3 *(HNC, Theorem 3.6, pg.63, [10]) The condensation network $\boldsymbol{g}^\star$ of any directed network $\boldsymbol{g}$ is acyclic.*

Let $M(\boldsymbol{g}^\star) = \{F(X) \in V(\boldsymbol{g}^\star) \mid$ there is no $F(Y) \in V(\boldsymbol{g}^\star)$ such that $F(Y) \succeq_{\boldsymbol{g}^\star} Fg(X)\}$ be the set of greatest elements of $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$. We use two lemmas given in Appendix to establish Proposition 4. The first one shows that the condensation network induced by a non-empty strict Nash network has a unique source. The second lemma establishes that in a condensation network induced by a strict Nash network, if a vertex x receives two arcs from y and z, then y and z receive no arc from another vertex.

The first part of the next proposition provides a general property which satisfies condensation networks induced by strict Nash networks. The second part of the proposition provides some properties about the architecture of the condensation networks induced by strict Nash networks.

Proposition 4 *Suppose the payoff function of each player i satisfies (1) and let $\boldsymbol{g}$ be a non-empty strict Nash network. Then,*

1. *$(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a inf-semi-lattice. Moreover, if $F(X) \in V(\boldsymbol{g}^\star)$ is not the minimal element of $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$, then $|X| = 1$.*

2. *$(V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a chain and for all $F(X), F(Y) \in M(\boldsymbol{g}^\star)$, we have $F(X) \wedge F(Y) = F(Z)$ where $F(Z)$ is the maximal element of $(V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$.*

Proof Let $\boldsymbol{g}$ be a non-empty strict Nash network. Let $\boldsymbol{g}^\star$ be the condensation network induced by $\boldsymbol{g}$. We prove successively the two parts of the Proposition. To avoid trivialities, we assume that $|V(\boldsymbol{g}^\star)| \geq 3$.[8]

[8]For $|V(\boldsymbol{g}^\star)| = 1$ the proposition has no meaning. For $|V(\boldsymbol{g}^\star)| = 2$ the proposition means that the network is connected. The connectivity of the network in such a case could be shown by using the same arguments as for the case $|V(\boldsymbol{g}^\star)| \geq 3$.

1. First, we show that $\succeq_{\boldsymbol{g}^\star}$ is a partial order over $\boldsymbol{g}^\star$. We know that $\succeq_{\boldsymbol{g}^\star}$ is transitive and reflexive over $V(\boldsymbol{g}^\star)$. We need to show that $\succeq_{\boldsymbol{g}^\star}$ is antisymmetric over $V(\boldsymbol{g}^\star)$. By Theorem 3, we know that $\boldsymbol{g}^\star$ is acyclic. Consequently, if $F(X), F(Y) \in V(\boldsymbol{g}^\star)$ and $F(X) \neq F(Y)$, then $F(X) \not\succeq_{\boldsymbol{g}^\star} F(Y)$ or $F(Y) \not\succeq_{\boldsymbol{g}^\star} F(X)$. That is $\succeq_{\boldsymbol{g}^\star}$ is antisymmetric over $V(\boldsymbol{g}^\star)$.
   Second, we need to show that for all $F(X)$, $F(Y)$ in $V(\boldsymbol{g}^\star)$, there exists $F(Z)$ such that $F(X) \succeq_{\boldsymbol{g}^\star} F(Z)$ and $F(Y) \succeq_{\boldsymbol{g}^\star} F(Z)$. We know by Lemma 1 that there is a unique source in $\boldsymbol{g}^\star$, say $F(Z)$. We have two properties (a) for all $F(X) \in V(\boldsymbol{g}^\star)$ $F(X) \succeq_{\boldsymbol{g}^\star} F(Z)$, and (b) $V(\boldsymbol{g}^\star)$ is a finite set. It follows that $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a inf-semi-lattice.

   We now show that if $F(X) \in V(\boldsymbol{g}^\star)$ is not the minimal element of $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$, then $|X| = 1$. To introduce a contradiction, suppose that $F(X) \in V(\boldsymbol{g}^\star)$ is not the lower bound over $V(\boldsymbol{g}^\star)$ and $|X| > 1$. Since $F(X)$ is not the minimal element and $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a inf-semi-lattice, there exists a vertex, say $F(Y) \in V(\boldsymbol{g}^\star)$, such that the arc $F(X)\ F(Y) \in A(\boldsymbol{g}^\star)$. Since $|X| > 1$ there are players $i, i' \in X$ such that $i\ i' \in A(\boldsymbol{g})$ and since $F(X)\ F(Y) \in A(\boldsymbol{g}^\star)$, there are players $i'' \in X$ and $j \in Y$ such that $i''\ j \in A(\boldsymbol{g})$. Without loss of generality, we assume that $i'' = i$. Since $\boldsymbol{g}^\star$ is acyclic, j does not obtain the resources of players in X. If player j forms an arc with i', then she obtains resources at least equal to the resources obtained by i due to the arc $i\ i'$ and incurs the same cost. It follows that j has an incentive to form the arc $j\ i'$ since i has an incentive to form an arc with i'. Therefore, j does not play a strict best response in $\boldsymbol{g}$ and $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

2. We now consider the second part of the Proposition.
   First, we establish that $(V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a chain. We know that $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a partial order. We need to show that $\succeq_{\boldsymbol{g}^\star}$ is total over $V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$, that is either $F(X) \succeq_{\boldsymbol{g}^\star} F(Y)$ or $F(X) \succeq_{\boldsymbol{g}^\star} F(Y)$, for all $F(X), F(Y) \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$. To introduce a contradiction suppose that $F(X) \not\succeq_{\boldsymbol{g}^\star} F(Y)$ and $F(Y) \not\succeq_{\boldsymbol{g}^\star} F(X)$. Since $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a inf-semi-lattice and there exist $F(X), F(Y) \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$ such that $F(X) \not\succeq_{\boldsymbol{g}^\star} F(Y)$ and $F(Y) \not\succeq_{\boldsymbol{g}^\star} F(X)$, there is a vertex in $\boldsymbol{g}^\star$ which has two predecessors. Let $K(\boldsymbol{g}^\star) = \{F(X) \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star) \mid$ there exist $F(Y), F(Z) \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star) : F(Y)\ F(X), F(Z)\ F(X) \in A(\boldsymbol{g}^\star)\}$ be the set of vertices in $V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$ which have two predecessors in $\boldsymbol{g}^\star$. Since $K(\boldsymbol{g}^\star)$ is finite and $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a inf-semi-lattice, $(K(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ admits a minimal element, say $F(X_0)$. Let $F(Y_0)$ and $F(Z_0)$ be two vertices such that $F(Y_0)\ F(X_0) \in A(\boldsymbol{g}^\star)$, and $F(Z_0)\ F(X_0) \in A(\boldsymbol{g}^\star)$. $F(Y_0)$ and $F(Z_0)$

cannot have an arc in $\boldsymbol{g}^\star$ with a vertex $F(X) \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$, $F(X) \neq F(X_0)$, otherwise either $K(\boldsymbol{g}^\star)$ is not a finite set which admits $F(X_0)$ as minimal element, or $(V(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is not a inf-semi-lattice. Moreover, since $F(Y_0) \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$ there exists $F(Y_1) \in V(\boldsymbol{g}^\star)$ such that the arc $F(Y_1)\ F(Y_0) \in A(\boldsymbol{g}^\star)$. Consequently, the assumptions of Lemma 2 are satisfied. It follows that $\boldsymbol{g}^\star$ is not a condensation network induced by a strict Nash network, a contradiction.

Second, we establish that for all $F(X), F(Y) \in M(\boldsymbol{g}^\star)$, we have $F(X) \wedge F(Y) = F(Z)$ where $F(Z)$ is the maximal element of $(V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$. We know that this maximal element exists. Indeed, $(V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a chain, so it contains a maximal element. Moreover, by Lemma 1 there exists a vertex in $V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$, say $F(Z)$, which is a source in $\boldsymbol{g}^\star$. Consequently, for each $F(X) \in M(\boldsymbol{g}^\star)$ there exists $F(X') \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$ such that the arc $F(X)\ F(X') \in A(\boldsymbol{g}^\star)$. We now show that $F(X), F(Y) \in M(\boldsymbol{g}^\star)$ have formed an arc with the same vertex $F(Z') \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$ in $\boldsymbol{g}^\star$. To introduce a contradiction, suppose that $F(X), F(Y) \in M(\boldsymbol{g}^\star)$ are such that $F(X)\ F(X') \in A(\boldsymbol{g}^\star)$ and $F(Y)\ F(Y') \in A(\boldsymbol{g}^\star)$ with $X' \neq Y'$. Clearly $F(X'), F(Y') \in V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)$. Moreover, since $(V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star), \succeq_{\boldsymbol{g}^\star})$ is a chain we can assume wlog that $F(X') \succeq_{\boldsymbol{g}^\star} F(Y')$. Hence there is a vertex $F(Y'') \in (V(\boldsymbol{g}^\star) \setminus M(\boldsymbol{g}^\star)) \setminus \{F(Y)\}$ such that $F(Y'')\ F(Y') \in A(\boldsymbol{g}^\star)$. It follows that there exist four vertices in $\boldsymbol{g}^\star$ which satisfy assumptions of Lemma 2. We conclude that $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

$\square$

The first part of Proposition 4 provides two insights. First, the condensation network induced by a strict Nash network is not always a lattice. In other words, there exist situations such that there are several players whose resources are not accessed by others. However, since the condensation network induced by a strict Nash network is a inf-semi-lattice, there is a class of player, say X, such that all players $j \in N$ obtain the resources of X in a non-empty strict Nash network. Secondly, if several players belong to the same equivalence class with regard to the set of resources obtained, then they do not obtain resources from players who do not belong to this class.

The second part of Proposition 4 highlights the existence of two sets of players. Players who belong to the first set, $M(\boldsymbol{g}^\star)$, are such that they obtain the resources of all players who do not belong to this set, but no player in the population accesses to the resources owned by these players. Players who belong to the second set, $N \setminus M(\boldsymbol{g}^\star)$, are such that each of them has a predecessor. Hence, there is a hierarchy between players who belong to this set.

Corollary 2 *Suppose the payoff function of each player i satisfies (1) and let $\boldsymbol{g}$ be a non-empty strict Nash network. Then, $\boldsymbol{g}^\star$ is a tail star.*

Proof Let $\boldsymbol{g}$ be a non-empty strict Nash network. Given Proposition 4 the Hasse diagram of $\boldsymbol{g}^\star$ is a tail star. □

Obviously, if $\boldsymbol{g}^\star$ contains 1 vertex, then $\boldsymbol{g}^\star$ is empty.
Let us now show through an example that there are situations such that the condensation network induced by a strict Nash network is a tail star.

Example 2 Suppose $N = \{1, 2, 3, 4\}$. Let $v_1 = v_2 = 1$, $v_3 = v_4 = 4$, $c_1 = c_2 = 7$, $c_3 = 1$, and $c_4 = 2$. We suppose that $F(\{1\}) = 1$, $F(\{2\}) = 2$ and $F(\{3, 4\}) = 5$, Straightforward computations show that the network $\boldsymbol{g}^\star$, drawn in Figure 2, is a condensation network induced by a strict Nash network.

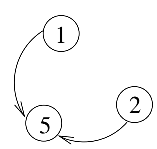

Figure 2: $\boldsymbol{g}^\star$ condensation network induced by a strict Nash network

We now compare the result given in Corollary 2 with the result obtained in the player heterogeneity framework by Galeotti (Proposition 3.1, pg.169, [9]). Recall that in the player heterogeneity framework, condensation networks induced by non-empty strict Nash networks are either empty, or center sponsored stars. Let us deal with situations where condensation networks induced by non-empty strict Nash networks are center sponsored stars. In such a star, $\boldsymbol{g}^\star$ there is one equivalence class which obtains the resources of all other equivalence classes while the latter obtain no resources from other equivalence classes. In other words, if the center sponsored stars contains x vertices, then there is a vertex, say $F(X_0)$, which obtains the resources of all other $x - 1$ vertices. Therefore, players in X_0 obtain resources of all other players. Moreover, it is not possible to compare the other vertices with respect to the relation $\succeq_{\boldsymbol{g}^\star}$. In the player heterogeneity model there is a dichotomy with regard to the resources obtained by players. Indeed, players in one equivalence class obtain resources from all other players while other players obtain only their own resources. By contrast, in condensation networks induced by strict Nash networks in partner heterogeneity model, this type of dichotomy where one player obtains all resources and the others not does not arise. Indeed, when the condensation network is a tail star, it is easy to see the resources sets of players form a gradual hierarchy.

Finally, costs do not play the same role in the player heterogeneity framework and in the partner heterogeneity framework. In the former, the results hold when the costs are homogeneous and when the costs are player heterogeneous are qualitatively equivalent. By contrast, in our framework the set of networks that are candidates to strict Nash is larger when the costs are partner heterogeneous than when the costs are homogeneous. Formally, a non-empty condensation network induced by a strict Nash network is a lattice, that is a inf-semi-lattice where each couple of vertices has a least upper bound,[9] when the costs are homogenous while it is a inf-semi-lattice when the costs are partner heterogeneous.

5 Conclusion

The most stable empirical finding concerning the structural properties that networks exhibit in reality is that networks have very asymmetric architectures. For instance, the WWW has very asymmetric architectures (see Barabási, Albert and Jeong [12]). This makes the study of the determinants of asymmetries crucial. In this paper, we establish that the nature of the heterogeneity plays an important role in the asymmetries observed in the equilibrium networks. To obtain our results, we use some new tools which allow to characterize the properties of the network with respect to the resources flow between players. These tools are particularly useful when both costs and values are partner heterogeneous. To the best of our knowledge this alternative approach has never been used in the network formation literature.

[9] A chain is a lattice.

6 Appendix

Lemma 1 *Suppose the payoff function of each player i satisfies (1). Let $\boldsymbol{g}^\star$ be the condensation network induced by $\boldsymbol{g}$, with $\boldsymbol{g}$ a non-empty strict Nash network. Then, $\boldsymbol{g}^\star$ has a unique source.*

Proof Let $\boldsymbol{g}$ be a non-empty strict Nash network. Let $\boldsymbol{g}^\star$ be the condensation network induced by $\boldsymbol{g}$.

First, we show that $\boldsymbol{g}$ is connected. Since $\boldsymbol{g}$ is non empty, there is a player, say j, who sponsors an arc, say ji, in $\boldsymbol{g}$. Moreover, $\boldsymbol{g}$ is a strict Nash network, hence we have $\pi_j(A_j(\boldsymbol{g}), A_{-j}(\boldsymbol{g})) - \pi_j(A_j(\boldsymbol{g}) - j\ i, A_{-j}(\boldsymbol{g})) > 0$. To introduce a contradiction, suppose that $\boldsymbol{g}$ is not connected. Then, there is a player i' such that $N_i(\boldsymbol{g}) \cap N_{i'}(\boldsymbol{g}) = \emptyset$. If player i' forms an arc with player i, then she obtains a marginal payoff equal to $\pi_{i'}(A_{i'}(\boldsymbol{g}) + i'i, A_{-i'}(\boldsymbol{g})) - \pi_{i'}(A_{i'}(\boldsymbol{g}), A_{-i'}(\boldsymbol{g})) \geq \pi_j(A_j(\boldsymbol{g}), A_{-j}(\boldsymbol{g})) - \pi_j(A_j(\boldsymbol{g}) - j\ i, A_{-j}(\boldsymbol{g})) > 0$. It follows that player i' does not play a strict best response in $\boldsymbol{g}$, a contradiction. Since $\boldsymbol{g}$ is connected, then $\boldsymbol{g}^\star$ is also connected.

Second, we show that there is a unique vertex which has formed no arc in $\boldsymbol{g}^\star$. To introduce a contradiction, suppose that there are two vertices, say $F(X)$ and $F(Y)$, which has formed no arc in $\boldsymbol{g}^\star$. Since $\boldsymbol{g}^\star$ is connected there is a player $i \notin X$ who has formed an arc with player $i' \in X$ in $\boldsymbol{g}$. Moreover, we have for each player $j' \in Y$, $N_{j'}(\boldsymbol{g}) \cap N_{i'}(\boldsymbol{g}) = \emptyset$. Since $\boldsymbol{g}$ is a strict Nash network we have $\pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) - \pi_i(A_i(\boldsymbol{g}) - i\ i', A_{-i}(\boldsymbol{g})) > 0$. If player $j' \in Y$ forms an arc with player i', then she obtains a marginal payoff equal to $\pi_{j'}(A_{j'}(\boldsymbol{g}) + j'i', A_{-j'}(\boldsymbol{g})) - \pi_{j'}(A_{j'}(\boldsymbol{g}), A_{-j'}(\boldsymbol{g})) \geq \pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) - \pi_i(A_i(\boldsymbol{g}) - i\ i', A_{-i}(\boldsymbol{g})) > 0$. It follows that player j' does not play a strict best response in $\boldsymbol{g}$ and $\boldsymbol{g}$ is not a strict Nash network, a contradiction. We conclude that $\boldsymbol{g}^\star$ has a unique source. $\square$

Lemma 2 *Suppose the payoff function of each player i satisfies (1). Let $\boldsymbol{g}^\star$ be the condensation network induced by $\boldsymbol{g}$, with $\boldsymbol{g}$ a non-empty strict Nash network. Then there do not exist $F(X_1), F(X_2), F(X_3), F(X_4) \in V(\boldsymbol{g}^\star)$ such that $F(X_2)F(X_1) \in A(\boldsymbol{g}^\star)$, $F(X_3)F(X_1) \in A(\boldsymbol{g}^\star)$ and $F(X_4)F(X_3) \in A(\boldsymbol{g}^\star)$.*

Proof Let $\boldsymbol{g}$ be a non-empty strict Nash network and let $\boldsymbol{g}^\star$ be the condensation network induced by $\boldsymbol{g}$. To introduce a contradiction, suppose that there exist $F(X_1), F(X_2), F(X_3), F(X_4) \in V(\boldsymbol{g}^\star)$ such that $F(X_1)F(X_2) \in A(\boldsymbol{g}^\star)$, $F(X_1)F(X_3) \in A(\boldsymbol{g}^\star)$ and $F(X_3)F(X_4) \in A(\boldsymbol{g}^\star)$. Let $\mathcal{Z} = \{F(X) \in V(\boldsymbol{g}^\star) :$ there exist $F(Y), F(Z) \in V(\boldsymbol{g}^\star)$ such that $F(Y)F(X) \in A(\boldsymbol{g}^\star)$ and $F(Z)F(X) \in A(\boldsymbol{g}^\star)\}$.

Since $\boldsymbol{g}^\star$ has a unique source, $\mathcal{Z}$ admits a unique element, say $F(Z_1)$, such that for all $F(Z') \in \mathcal{Z}$ we have $F(Z') \succeq_{\boldsymbol{g}^\star} F(Z)$. We consider $F(Z_2), F(Z_3), F(Z_4) \in V(\boldsymbol{g}^\star)$ such that $F(Z_2)F(Z_1) \in A(\boldsymbol{g}^\star)$, $F(Z_3)F(Z_1) \in A(\boldsymbol{g}^\star)$ and $F(Z_4)F(Z_3) \in A(\boldsymbol{g}^\star)$.

First, we show that $F(Z_3)$ sponsors a unique arc in $\boldsymbol{g}^\star$. To introduce a contradiction, suppose that $F(Z_3)$ forms an arc with a vertex $F(Z_5)$ in $\boldsymbol{g}^\star$. Since $\boldsymbol{g}^\star$ has a unique source, say $F(X')$, we have $F(Z_5) \succeq_{\boldsymbol{g}^\star} F(X')$. Hence there is a path, say P, from $F(Z_5)$ to $F(X')$ in $\boldsymbol{g}^\star$. There exist two cases, either P goes through $F(Z_1)$, or not. If P goes through $F(Z_1)$, then the BSNP is not satisfied. If P does not go through $F(Z_1)$, then there exists a vertex in $\mathcal{Z}$, say $F(Y')$, such that $F(Z_1) \succeq_{\boldsymbol{g}^\star} F(Y')$ a contradiction.

It follows that there exist players $i_1 \in Z_1$, $i_2 \in Z_2$, $i_3 \in Z_3$, $i_4 \in Z_4$ such that $i_2 i_1 \in A(\boldsymbol{g})$, $i_3 i_1 \in A(\boldsymbol{g})$ $i_4 i_3 \in A(\boldsymbol{g})$. Moreover, since $F(Z_3)$ forms a unique arc in $\boldsymbol{g}^\star$, we have: $\pi_{i_4}(A(\boldsymbol{g})) - \pi_{i_4}(A(\boldsymbol{g}) - i_4 i_3 + i_4 i_1) = \sum_{\ell \in Z_3} v_\ell - c_{i_3} > 0$. The inequality is due to the fact that $\boldsymbol{g}$ is a strict Nash network. Likewise, we have $\pi_{i_2}(A(\boldsymbol{g})) - \pi_{i_2}(A(\boldsymbol{g}) - i_2 i_1 + i_2 i_3) = \sum_{\ell \in Z_3} v_\ell - c_{i_3} < 0$. The inequality is due to the fact that $\boldsymbol{g}$ is a strict Nash network. The two inequalities are not compatible together, a contradiction. $\square$

References

[1] Venkatesh Bala and Sanjeev Goyal. A noncooperative model of network formation. *Econometrica*, 68(5):1181–1230, September 2000.

[2] Matthew O. Jackson. *Social and Economic Networks*. Princeton University Press, August 2008.

[3] M. E. J. Newman. The Structure and Function of Complex Networks. *SIAM Review*, 45(2):167–256, 2003.

[4] Robert P. Gilles and Cathleen Johnson. Spatial social networks. *Review of Economic Design*, 5(3):273–299, 2000.

[5] Michael McBride. Position-specific information in social networks: Are you connected? *Mathematical Social Sciences*, 56(2):283–295, September 2008.

[6] Hans Haller and Sudipta Sarangi. Nash networks with heterogeneous links. *Mathematical Social Sciences*, 50(2):181–201, September 2005.

[7] Daniel A. Hojman and Adam Szeidl. Core and periphery in networks. *Journal of Economic Theory*, 139(1):295–309, March 2008.

[8] Pascal Billand, Christophe Bravard, and Sudipta Sarangi. Strict Nash networks and partner heterogeneity. *International Journal of Game Theory*, 40(3):515–525, August 2011.

[9] Andrea Galeotti. One-way flow networks: The role of heterogeneity. *Economic Theory*, 29(1):163–179, September 2006.

[10] Frank Harary, Robert Z. Norman, and Dorwin Cartwright. *Structural Models: An Introduction to the Theory of Directed Graphs*. John Wiley & Sons Inc, New York, 1965.

[11] Garrett Birkhoff. Lattice theory. volume 25. American Mathematical Society, 1993.

[12] Albert-László Barabási, Réka Albert, and Hawoong Jeong. Scale-free characteristics of random networks: the topology of the world-wide web. *Physica A: Statistical Mechanics and its Applications*, 281(1-4):69 – 77, 2000.