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Dan Kovenock
Chapman University

Sudipta Sarangi
Louisiana State University

Matt Wiser
Louisiana State University

Working Paper 2012-06
http://bus.lsu.edu/McMillin/Working_Papers/pap12_06.pdf

Department of Economics
Louisiana State University
Baton Rouge, LA 70803-6306
http://www.bus.lsu.edu/economics/

All-Pay Hex: A Multibattle Contest With Complementarities*

Dan Kovenock † Sudipta Sarangi ‡ Matt Wiser §

November 13, 2012

Abstract

In this paper, we examine a modified 2 × 2 game of Hex in which control of each cell is determined by a Tullock contest. The player establishing a path of cells within his control between his two sides wins a fixed prize. Examining the polar cases of all cells being contested simultaneously versus all four cells being contested sequentially, we show that there is an increase in the total expected payoff for the players in the sequential case compared to the simultaneous case. Furthermore, due to the players having different, albeit symmetric winning combinations, in the sequential case one player may have a greater expected payoff than their opponent, which depends on the order of the cell contests. We thus provide a canonical model of a multibattle contest in which complementarities between battlefields are heterogeneous across both battlefields and players.

Keywords: Contests, All-Pay Auctions, Multibattle, Complementarity, Hex

JEL Classification: C72, C73, D72, D74

*We would like to thank Kalyan Chatterjee, Cary Deck, and Brian Roberson for their suggestions, as well as participants at the *Tournaments, Contests and Relative Performance Evaluation* (2011) conference at North Carolina State University and the *Public Economic Theory* (2011) conference at Indiana University for their comments. We would also like to thank two anonymous referees and the Associate Editor for useful comments.

†Department of Economics, Chapman University. Email: kovenock@chapman.edu

‡Department of Economics, Louisiana State University. Email: sarangi@lsu.edu

§Department of Economics, Louisiana State University Email: mwiser1@lsu.edu

1 Introduction

We examine a competition comprised of multiple contests, combinations of which exhibit complementarity. In our game a benefit accrues to a player only by having won one of several winning combinations of contests. Players have different winning combinations, reflecting differing goals, but the combinations are such that exactly one player will always win. The complementarity between contests arises because success in a single contest or set of contests may yield the same payoff as losing every contest but combined with one more contest win may yield overall victory. These factors result in variations in the valuation of individual contests based on the identity and outcomes of contests already decided and the order of contests yet to be played.

The basic structure for this competition is given by the board game Hex. In the canonical form, Hex is played by two players on a 11 × 11 grid of hexagonal cells. The players are conventionally labeled Black and White. Each player alternates claiming an unclaimed cell of the board. Black's objective is to connect the two black sides of the board with a path of his pieces, while White's is to connect the two white sides of the board with her pieces. The figure below shows a game in progress (Weisstein, 2010).

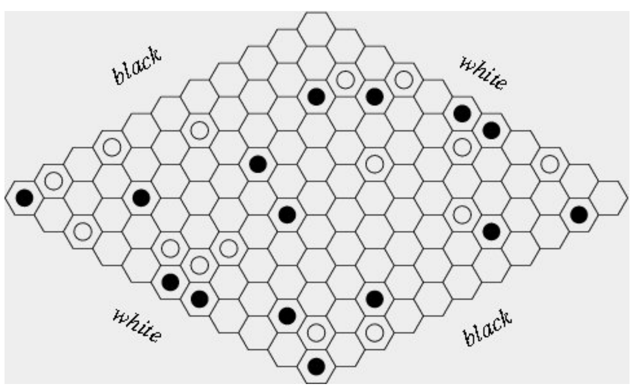

Figure 1: **Game of Hex in Progress** (Weisstein, 2010)

As the game continues, the player who is able to connect her two assigned sides is declared the winner. Draws are impossible. If it becomes impossible for one player to make a connection, it means that all of their routes have been cut off. This in turn implies that there must be a continuous path that wins for the opposing player. We will use this basic structure. However, for the sake of tractability in the multibattle context, we will reduce the grid to being 2 hexagons on each side, as opposed to 11. Also, for simplicity, each player will value the prize that is being contested identically.

Network security provides a good practical example of this type of structure, because data can be routed around compromised servers as long as a connection exists between two nodes, thus avoiding servers which have been hacked or damaged. However, intermediate relay nodes hold no intrinsic value, since value is entirely due to the final connection. Thus we do not need to worry about the value of individual cells, only the completed path. These networks are often geographically dependent, such as wireless relay towers and fibre optic switching stations. Another example of a similar network is a cellular phone system, where towers relay signals, and interference limits the number of towers in an area.

Related work can be found in the literature on Colonel Blotto games (Borel 1921, Borel and Ville 1938, Gross 1950, Gross and Wagner 1950, Friedman 1958).[1] These games feature two players who simultaneously allocate their respective fixed budgets of a resource across n different contests, with the higher allocation in a given contest winning the contest. Players choose their allocations to maximize the expected number of battlefields won. In these early papers linkages between contests arise from the budget constraints; allocation of a unit of the resource to one contest reduces the availability of the resource to other contests. Recently there has been a resurgence of interest in these games with extensions to the cases of asymmetric budgets and a positive opportunity cost of the resource in games with both continuous and discrete strategy spaces (Hart 2008, Kvasov 2007, Laslier 2002, Laslier and Picard 2002, Roberson 2006, and Weinstein 2005). Colonel Blotto games have also been examined under the assumption that the winner of each contest is determined probabilistically by the players respective allocations according to a Tullock contest success function (Tullock, 1980), with the success function itself having been introduced previously (Tullock, 1975). Contributions employing the Tullock contest success function include Friedman (1958) and Robson (2005).

In addition to linkages between contests that arise through the cost of resource allocation, such as the individual budget constraints of the Colonel Blotto game, there are also linkages that arise through the way in which individual battlefield outcomes are aggregated in determining the players payoffs. Szentes and Rosenthal (2003 a,b) examine a game in which players simultaneously allocate a resource at constant unit cost to n different battlefields. A player earns a prize of common and known value if he is the higher bidder in m of those battlefields. The special case where n is odd and $m = \frac{n+1}{2}$ is the game in which the player who wins a majority of the contests is the winner. Szentes and Rosenthal solved this game for $n = 3$, but for $n > 3$ this remains an open problem.

[1] See Kovenock and Roberson 2010a for a survey of these and related games.

The corresponding n battlefield majoritarian problem with a generalization of the Tullock contest success function with exponent $\alpha \leq 1$ was examined by Snyder (1989) who obtained some partial results. Klumpp and Polborn (2006) solved the n battlefield game for a Tullock contest success function with exponent $\alpha \leq 1$ more generally, characterizing the nature of the nondegenerate mixed strategy equilibria that arise when there are sufficiently many battlefields that no pure strategy equilibrium exists.

More complex linkages that arise from the way in which battlefield outcomes are aggregated have also been examined. Clark and Konrad (2007) examine a game with n battlefields, a constant unit cost of expenditure, and a Tullock contest success function with exponent $\alpha = 1$, in which one player must win all of the contests in order to win a prize while the other needs only win at least one contest. Kovenock and Roberson (2010b) examine the corresponding game under the assumption that the high bidder in each contest wins the contest. Golman and Page (2009) examine a modified Colonel Blotto game, which they term "General Blotto," that takes the original budget-constrained Colonel Blotto game in which the high bidder wins each contest and adds compound contests formed by taking subsets of the sets of battlefields. In each of these added compound contests, a players allocation is taken to be the product of the allocations in the battlefields defining the compound contest and the player with the higher such product wins. These contest wins are then added to those of the single battlefields to determine the number of contest won.

Our model is similar in spirit to the models with payoffs determined by a nonlinear aggregation of battlefield outcomes. In our model players allocate resources at a constant and identical unit cost to four battlefields. In the main text we analyze the polar cases with all cells being contested simultaneously and sequentially, with the sequential case done both when the order is known a priori and when it is random. Intermediate cases are also solved in the Appendices. Each battlefield outcome is determined by a Tullock contest success function with $r = 1$ (the lottery contest success function). The battlefields are spatially distributed to correspond with cells in a 2×2 game of Hex and the overall contest winner is the player who wins a configuration of contests that would win the game of Hex. Consequently, multiple combinations of individual contest wins may earn a player the overall prize, but exactly one player will be a winner. Players are risk neutral so that they maximize the expected prize winnings minus the cost of their allocation to the four battlefields.

The paper is laid out as follows. First, an example is described in Section 3, showing the general method of play. The model is then formally defined in Section 4, and specific illustrative examples detailed in Section 5. Finally, some robust conclusions about the game in general are presented in

Section 6, with the mathematical details of the cases in the Appendices.

2 A Brief History of Hex

Hex occupies an unusual place in the history of game theory for several reasons. First, one of the two independent inventors of the game in the 1940's was John Nash, while the other was the Danish mathematician Piet Hein. Secondly, this game was proven to have a first mover advantage well before a winning strategy itself was found, even for smaller cases. The proof uses the notion of the strategy stealing argument and proceeds by contradiction. Note that in this game owning a space is always beneficial. Now if the second player had a winning strategy, the first player could copy this winning strategy with the advantage of already having a space. As ties have been ruled out, this guarantees a win for the first player (Berlekamp, 1982).

Hexagons are used instead of squares, as in a square gird, there are pairs of cells which adjoin only at corners. For example, in a checkerboard, the dark squares are only adjacent to light squares on the edges, however the dark squares touch other dark squares at their corners. A checkerboard also illustrates the problem with corner adjacency, as if touching corners are considered adjacent, light squares and dark squares both form paths across the board. However, if corners do not establish adjacency, neither light nor dark squares have a path across the checkerboard. As hexagonal cells never touch at only a corner, this issue does not arise for hexagonal grids, and thus in the end, exactly one player will complete a winning path.

Furthermore, the complexity of this game has proven very difficult (Evan et al., 1976), since determining an optimal strategy is PSPACE-Complete. PSPACE-Complete problems are those where obtaining a solution would require an ideal computer to have memory proportional to a polynomial function of the size of the problem.[2] The size of the problem in the case of Hex would be the size of the grid being played on. Thus the standard game of Hex would require a very large amount of memory to be certain that a perfect strategy has been found, and thus far such a solution remains unknown. Work continues on this game as in Anshelevich (2000) and Campbell (2004), including proposals for computer players employed in the absence of a solution for the general case.

[2]PSPACE-Complete problems are considered to be unsolvable in an amount of time equal to a polynomial function of the size of the problem, however this is not proven.

3 All Pay Hex: An Illustrative Example

The structure of Hex provides a useful starting point for examining complementarities. Given each player's goal of connecting the opposite sides, winning combinations will vary, and the value of a cell will depend on the use of this cell in creating a winning path. Since the canonical version of Hex is computationally difficult, we will focus on a tractable 2 $\times$ 2 version with four cells being contested.

Our game differs from canonical Hex in that the players play simultaneously. Each player simultaneously commits a resource to the cells currently being contested. The winner of each cell is determined by the Tullock contest success function, given by $P(X) = \frac{X}{X+Y}$, where X and Y are the amounts committed to the cell being contested. After a set of cells has been contested, players observe whether either player has won the game, thereby receiving a prize V. If not, the outcomes are observed and a new subset of cells is contested, repeating this process until a winner of the overall contest is found. These subsets of cells come from taking an ordered partition of the set of cells, with the subset being contested in round n being the elements in the nth subset of the partition.

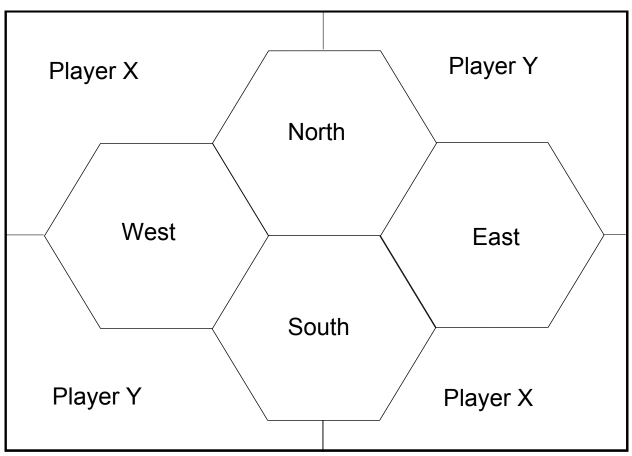

Figure 2: **The Cells Being Contested**

For the sake of illustration, let the prize for completing a connection be 100, with players simultaneously competing for all four cells. Suppose player X invests 20 in both the North and East cells, and only 1 in West and South, in an effort to have overwhelming force in 2 cells. Meanwhile, Player Y invests 10 in both the North and South, and 5 in the East and West.

We can now calculate the probabilities of X winning each cell. Based on the contest success function, in the North, she has a $\frac{2}{3}$ chance of victory, in the East $\frac{4}{5}$, in the West $\frac{1}{6}$, and in the South $\frac{1}{11}$. Thus player X has a chance of victory given by the sum of his probabilities of winning both the North and East $\frac{2}{3} \cdot \frac{4}{5}$, both the North and South but not the East $\frac{2}{3} \cdot \frac{1}{11} \cdot \frac{1}{5}$, and both the West and South but not the North $\frac{1}{6} \cdot \frac{1}{11} \cdot \frac{1}{3}$. This covers all winning sets for player X without double counting any sets, as all eight winning sets fall into exactly one of these three possibilities.

This gives a total chance of victory for player X of approximately 0.5505, i.e. his expected earnings are 55.05 at a cost of 42, for a net gain of 13.05. Player Y will have a 0.4495 probability of winning, and thus expected winnings of 44.95. However, player Y will have a net gain of 14.95, as she spent only 30.

4 The Model

We will now develop our formal model of the 2×2 case of All-Pay Hex. The game is played over the set of cells $A = \{N, S, E, W\}$.

Players: We denote the two risk neutral players in the game by X and Y. It is assumed that the players do not have a budget constraint.

Strategies: Since there are four cells, we will allow for the possibility that they can be contested at different points in time. Before the contest begins, the contest structure R is announced.[3] Let $R = \{R_1, R_2, R_3, R_4\}$ be an ordered partition of A, where R_r is the set of cells being put up for contest in round r and let $|R_r| = C_r$. Since there are only four cells, we assume that the number of rounds cannot exceed this number.

In each round r, each player chooses an C_r-vector of bids with the bid for cell i by player $T \in \{X, Y\}$ being labeled T_i. We will require $T_i \geq 0$ and define $Z_i = \sum_{T=X,Y} T_i$.[4] The winner of each cell will be determined by the Tullock contest function, so that player T will win cell i with probability $\frac{T_i}{Z_i}$.

Payoffs: The payoff function of each player takes into account the expected benefits minus the costs. Each player obtains a identical benefit V from winning the game. We will require some

[3] We also explore the implications of relaxing this assumption and having a random sequence later in the paper. We thank an anonymous referee for suggesting this.

[4] $T_i = 0$ will be limited to cases in which cell i has been made irrelevant by earlier rounds. This is a technical condition imposed by the Tullock contest success function.

additional notation for our calculations. Let $X^\bigstar$ be the collection of winning sets of cells for player X, and $Y^\bigstar$ for player Y. Specifically, from figure 2 it can be seen that

$X^\bigstar = \{\{N,E\},\{N,S\},\{S,W\},\{N,S,E\},$ $\{N,S,W\},\{N,E,W\},\{S,E,W\},\{N,S,E,W\}\}$	$Y^\bigstar = \{\{N,W\},\{N,S\},\{S,E\},\{N,S,E\},$ $\{N,S,W\},\{N,E,W\},\{S,E,W\},\{N,S,E,W\}\}$

Note that each player has three minimal winning sets consisting of two cells, along with the supersets of these. The set $\{N,S\}$ is the common minimal winning set for either player. Thus the probability that player X wins the prize is

$$\sum_{\alpha \in X^\bigstar} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \right) \tag{1}$$

Hence the payoff of player X can be written as

$$U_X \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R \right) = \sum_{\alpha \in X^\bigstar} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \right) V - \sum_{i \in A} X_i \tag{2}$$

where the second term is the amount of money that X spends. Note that the utility function for player X is a function of the amounts invested in each cell, which will be influenced by the round structure Σ. Thus we take the amounts invested by each player in the first round, the amounts invested in the second round, and so on until all cells have been accounted for. Similarly, Player Y's payoff function is given by:

$$U_Y \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R \right) = \sum_{\alpha \in Y^\bigstar} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{Y_i}{Z_i} \frac{X_j}{Z_j} \right) V - \sum_{i \in A} Y_i \tag{3}$$

5 Solving the Game

Before we analyze the game, we quickly discuss what happens when there are no complementarities between the cells. If there are no complementarities, the values of the cells are independent of one another. Then, if V_N is the common value to the players of winning the North cell, Player X's expected payoff for the contest occurring in the North will be $V_N \left(\frac{X_N}{Z_N} \right) - X_N$, and Player Y's expected payoff will be $V_N \left(\frac{Y_N}{Z_N} \right) - Y_N$. From the first order conditions we have $V_N \left(\frac{Y_N}{Z_N^2} \right) = V_N \left(\frac{X_N}{Z_N^2} \right) = 0$, which gives us $X_N = Y_N = \frac{V_N}{4}$. Corresponding calculations apply to the East, South, and West cells. Thus each player will spend a quarter of the value of each cell, and have an expected payoff of a quarter of the value of the cell.[5]

From the previous section it should be clear that a complete analysis of the problem requires examining several different cases to identify the effects of the complementarity involved. The role of complementarity also varies depending on whether the winning sets are contested separately or simultaneously. Hence we will focus on two polar cases: those of all cells being contested simultaneously and the four cells being contested sequentially. The intermediate cases, where cells are contested over 2 or 3 rounds, will be examined briefly. Full solutions are provided in Appendix B and summarized in Table 2. The main distinction between the simultaneous and sequential cases is that the sequencing of cells allows for the introduction of asymmetries which are not otherwise possible. While there is only one case when all four cells are contested simultaneously, there are multiple distinct subcases for sequential contests, depending on the order of the cells. In the remainder of the paper we normalize V to 1.

I. Simultaneous Contests

To solve this game, we will simultaneously maximize the expected payoff of each player given by equations 2 and 3. Since the payoff functions are concave in each player's own allocation, this gives us a unique Nash equilibrium.[6] First, observe that there is a great deal of symmetry in the problem, with East for X being strategically equivalent to West for Y, North for X being strategically equivalent to South for Y. Moreover East is strategically equivalent to West for each player, as are North and South.[7] This creates symmetric payoff functions for the two players, leading to symmetric strategies in equilibrium. Thus, our problem only has two variables. The remark below summarizes the main findings for the simultaneous case, and the equilibrium calculations are provided in Appendix A.

Remark 1: When all four cells are contested simultaneously, in the unique Nash equilibrium both players will employ a symmetric strategy of spending $\frac{1}{8}$ on each of the North and South cells, and spending $\frac{1}{16}$ on each of the East and West cells, giving each player a probability of victory of $\frac{1}{2}$ and an expected payoff of $\frac{1}{8}$.

[5]Note that this differs from the typical Colonel Blotto problem in the sense that players do not have to win a majority of the cells, rather they would like to win as many cells as possible since each cell provides benefits for winning. Moreover, they do not have to distribute limited total resources amongst the cells.

[6]This is shown in Appendix A.

[7]See Appendix A for more on this.

II. Sequential Contests with Pre-Specified Order

We now analyze sequential contests in which the order of play is known to both players before the contest begins. In order to solve for subgame perfect equilibrium strategies, we need to backward induct in the extensive form of the game, and obtain expected payoffs for all possible sequences in the final stage. These are then used in determining expected payoffs in the previous round, and so on to obtain the subgame perfect equilibrium. Given that the two players have differing winning sets, the ordering of cells can bias the game in favor of one player, as winning sets can become available to players at different times. In all cases where either North or South is the first cell contested, the game will still be symmetric between players X and Y. This is not necessarily true if East or West is the first cell contested. In these situations, if the first two cells contested form a winning set for one player, then there will be a bias in favor of the other player. For example, W-N-E-S induces a bias in favor of X. If they do not form a winning set then the game still remains symmetric, for example W-E-N-S has no bias towards either player. This is explained in detail later.

We now compare the two polar cases in terms of underdissipation. Underdissipation is the situation where the total aggregate expenditure of the players is less than the value of the prize being contested.[8]

Proposition 1. *All sequential structures have lower expected dissipation than the simultaneous case.*

Proof: See Appendix A.

The table below summarizes the payoffs of both players for the two polar cases. Detailed calculations of these results can be found in Appendix A.

Type	**Order**	$E[U_X]$	$E[U_Y]$
4	Simultaneous (NESW)	.125	.125
1-1-1-1	N or S as first round	.1797	.1797
1-1-1-1	E-W or W-E as first and second rounds	.1406	.1406
1-1-1-1	E-N or W-S as first and second rounds	.0731	.2315
1-1-1-1	W-N or E-S as first and second rounds	.2315	.0731

Table 1: Expected Payoffs under Simultaneous and Sequential Structures

[8]Similarly, overdissipation occurs when players spend more in the aggregate than the common value of the prize. The possibility of overdissipation in a game with Tullock contest functions has been explored, for instance in (Baye et al., 1999).

Before proceeding further, some explanation of the table is in order. In the table, Type refers to the number of cells contested in each round, and Order denotes which cells are contested in each round. In each case, the rounds are separated by dashes. In the third row, the first line consists of those sequential structures where North or South is the first cell, and the second line are those with East and West as the first two rounds, with North and South in either order in the third and fourth rounds. The last row can be understood in a similar manner. $E[U_X]$ and $E[U_Y]$ are the expected benefits of players X and Y respectively.

Remark 2: In a four round contest, the ordering of the *two* cells to be contested in the third and fourth rounds is irrelevant. There are three possible cases after the first two rounds: (i) either one player has won, or (ii) only one cell is relevant, in which case the other cell can be ignored, or (iii) one player must win both remaining cells. In the first two cases, as we can ignore irrelevant cells the joint ordering cannot matter. In the third case, one player must win both remaining cells and the order in which the cells are contested is irrelevant. Thus, there is no situation under which the ordering of the third and fourth rounds matter. Note that an intermediate case with only 3 rounds, such as E-W-NS, where East is the first round, West the second round and North and South are to be contested simultaneously in the third round, is not equivalent.[9]

Intuitively, underdissipation occurs in the sequential case due to asymmetries between players in the number of winning subsets that have been covered by the contested cells at the conclusion of each round. Using the knowledge of which cells have been won by which player, the remaining options are spread across a few possible outcomes of remaining cells, allowing for better decisions. The sequencing creates these asymmetries, resulting in underdissipation. Our next two results identify conditions under which expected payoffs in the sequential case are asymmetric and symmetric respectively.

Proposition 2. *In the sequential case, asymmetric expected payoffs between players X and Y require that the cells in* $\{R_1, R_2\}$ *form an element of exactly one of* $\{X^\star, Y^\star\}$*. If* $\{R_1, R_2\}$ *is an element of* $X^\star$*, then EV[X]* $\leq$ *EV[Y] for the entire game, while if* $\{R_1, R_2\}$ *is an element of* $Y^\star$*, EV[Y]* $\leq$ *EV[X] for the entire game.*

Proof: See Appendix A.

This proof proceeds by contradiction. While the formal proof is in Appendix A, here we provide a quick sketch. If North and South constitute the first two rounds, they form a winning set for *both*

[9] This is shown in Appendix B, and is due to the possibility that both North and South are still relevant after the second round.

players, and thus no asymmetry exists. Similarly, if East and West are the first two rounds, they form a winning set for neither player, and again, no asymmetry exists. In the remaining cases, one of North and South, and one of East and West constitute the first two rounds. Each combination of one of North and South and one of East and West form a minimal winning set for exactly one player.

We will now consider two distinct examples of sequential structures. This will illustrate the intuition behind this result, as well as demonstrate the fact that this Proposition provides a necessary but not a sufficient condition for payoffs to be asymmetric.

Since players have different winning sets, it may be possible for one player to obtain a winning set in a round in which the opposing player could not have done so. For example, consider the structure E-N-W-S. After the second round $r = 2$, the set of contested cells is $\{E, N\}$, an element of $X^{\star}$. In this case player X has an expected payoff of approximately 0.0731 before the first round, while player Y has an expected payoff of approximately 0.2315, as shown in Appendix A.[10] Observe that in the E-N-W-S structure, after the second round, player X could have a winning set, while this is not possible for player Y. Note that if the player who could have completed a winning set has failed to do so, this player has fewer possible winning sets available to complete in subsequent rounds.

Following our example, if player X has not won after the second round he could not have won *both* the North and East, eliminating his winning sets $\{\{N, E\}, \{N, E, S\}, \{N, E, W\}, \{N, E, S, W\}\}$ from consideration, leaving $\{\{N, S\}, \{S, W\}, \{N, S, W\}, \{E, S, W\}\}$. No such eliminations are possible for player Y, as even winning both the North and East does not comprise a winning set for Y. After a given round has been contested, if neither player has won, both players will be able to use the results of the contests up to that point for determining optimal strategies for the remaining rounds. The knowledge of what winning sets are still possible allows for recalculation of the value of each remaining cell by each player. If there is an asymmetry in the winning combinations that can still be possibly completed by the two players, they will have different opportunities to make use of this information. Thus, the knowledge of previous results may be more useful to one player, allowing her to obtain a greater expected payoff. Such asymmetries cannot occur in the simultaneous case, and so the expected payoffs must be identical.

Intuitively, two different factors are involved in the sequential case. Being able to form a

[10]This expected payoff for X is the lowest value that exists for a single player in any structure R. The equivalent sequence W-N-E-S provides the lowest expected payoff for Y which is identical.

winning set earlier provides a benefit from the greater chance of an early victory, thus ending the contest with spending in fewer rounds. In contrast, having a greater number of winning sets still available after a round allows a player to take advantage of the knowledge of the asymmetry in player strengths to increase his payoff.

In some situations these factors can balance each other out. For example, the structure N-W-S-E results in identical expected payoffs of approximately 0.1797 for each player, as shown in Appendix A. In this case, if after the second round, player Y has not yet won, then the winning sets $\{\{N,W\},\{N,S,W\},\{N,W,E\},\{N,S,E,W\}\}$ can no longer be obtained by player Y. However, player X does not have any winning sets necessarily eliminated after the second round. As player X would not win with winning both the North and West, all we know is that player X must have won one of the two cells contested. In spite of this difference, the expected payoffs for the entire game are the same for each player and we find that the two factors cancel each other out.

The fact that these two forces balance each other perfectly in this case may be due to the fact that although only Player Y has the possibility of victory after the second round, if Player X wins the North, the second round of West becomes irrelevant, and thus Player X can win after the third round (which is only the second round in which players actually expend effort). Thus both players have the ability to win after only two rounds of actual expenditures.

Thus Proposition 2 states only a necessary condition, not a sufficient one. In the sequential cases, the impact of this asymmetry favors the player that does not have a winning set available in the first two rounds. Note that under the intermediate cases, the asymmetry can favor either player. In some intermediate cases, the player with the advantage is the one who has a greater number of winning sets in which all cells have been contested at the end of round r. This can occur because the advantage of winning early may be greater than the advantage of having more winning sets available after the winners of some subset of cells have been determined

Proposition 3. *If a contest structure contains a round R_i, such that either $\{N,S\} \subseteq R_i$ or $\{E,W\} \subseteq R_i$, then the players have the same expected payoffs.*

Proof: See Table 2 and Appendices A and B.

We can see this result from the presentation of the expected payoffs of the players in Table 2. The forces that lead to Proposition 3 are similar to those driving Proposition 2. This proposition states that the players will exhibit symmetric behavior regardless of the round in which the $\{N,S\}$ or $\{E,W\}$ subset appears. Moreover, R_i prevents the existence of a winning set in another round for one player. Also note that this is a sufficient condition for identical expected payoffs, not a necessary one. For example, the order N-E-S-W results in identical expected payoffs for the players, despite lacking such a round.

5.1 Contests with Random Order

We will now consider a sequential contest in which the order of rounds is chosen randomly. Thus, unlike the previous section, in any given round the agents will not know the order in which the remaining cells will be contested in future rounds. First the cell to be contested in Round One. After the players make their Round One decisions and the outcome is realized, the cell to be contested in Round Two is announced. Again, the players make their Round Two decisions and only after the realization occurs is the Round Three cell announced. This process continues until all the cells are announced. Of course, given that there are only 4 rounds, once Round Three is over both players will know which cell remains to be contested, so the uncertainty is over the first 3 rounds (Remark 2). Observe that there are 24 possible random permutations of North, South, East, and West. We begin by splitting these into cases based on the first round.

Case 1: North or South as Round One

In 12 of the 24 possible orders, North or South will be the randomly drawn in the first round. Without loss of generality let North be randomly chosen first. If player X wins North, player X needs to win one of South and East to form a winning set, while if Player Y wins North, player Y must win one of South and West to form a winning set. This means that the winner of North must win one of two cells to be contested sequentially, while the loser of North must win two cells contested sequentially.[11] Thus winning the North gives a player an expected payoff of $\frac{43}{64}$ for the subgame consisting of East, West, and South, while losing the North gives a player an expected payoff of $\frac{1}{64}$ for these remaining three rounds (as obtained by backwards induction in Appendix A). Thus in Round One we have the expected payoff equations $U_X = \left(\frac{X_N}{Z_N}\right)\frac{43}{64} + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$ and $U_Y = \left(\frac{Y_N}{Z_N}\right)\frac{43}{64} + \left(\frac{X_N}{Z_N}\right)\frac{1}{64} - Y_N$. This gives us the first order equations $\left(\frac{Y_N}{Z_N^2}\right)\frac{42}{64} = \left(\frac{X_N}{Z_N^2}\right)\frac{42}{64} = 1$, which yield $X_N = Y_N = \frac{21}{128}$, and thus $U_X = U_Y = \frac{23}{128} \approx .1797$.

Note that this is the same as the expected payoff in the sequential case where North or South is the first cell contested. Once the first round is resolved, one remaining cell becomes irrelevant, effectively leaving a two round game. The player who lost the first round must win both of the

[11] The third remaining cell has no impact on the overall winner, and thus can be ignored.

remaining relevant rounds, thus making the ordering of the two still relevant cells irrelevant. This holds true both when the sequence is known in advance and when it is randomly drawn in each round, thus the identical expected payoffs follow directly.

Case 2: East or West as Round One

In 12 of the 24 possible orders, East or West will be the cell randomly drawn first. In these cases the ordering of the subsequent rounds matters. Thus, we must find expected payoffs for each player for all three possible second round contests. We then take the average across these three expected payoffs to find a total expected payoff. We then will then use these average expected payoffs to find the optimal strategy and the resulting expected payoff for the first round. Without loss of generality let East be randomly drawn in Round One.

Fourth Round: For the final round, either the cell will determine the overall winner, in which case each player will expend $\frac{1}{4}$, and thus have an expected payoff of $\frac{1}{4}$, or the cell is irrelevant and will not be contested.

Third Round: Moving backwards to the third round, once the cell being contested in the third round is announced, the cell to be contested in the fourth round is known by process of elimination. Thus the optimal strategies and expected payoffs for each player for the third round can be taken from the optimal strategies and expected payoffs found while obtaining the solutions for the sequential case (see Appendix A, case 2).[12]

Second Round: Assuming Player X won the East, whether the second round is West or South yields the same expected outcomes. This is because if X also wins the second round in either case, she will require one of the remaining two cells. As we saw in the sequential cases, at the start of this second round, X has an expected payoff of approximately 0.4245, and Y has an expected payoff of approximately 0.0455.

If the second round is North, the players have different expected payoffs, since X is victorious if he wins North. If X loses the North, X must win both the South and West. Again, taking the payoff from the fixed sequential case already solved, we see that at this stage X has an expected payoff of 0.2373, and Y of 0.1106.

First Round: Taking the average across all three possible second rounds gives a total expected payoff for X of $\frac{1}{3}(0.4245 + 0.4245 + 0.2373) = 0.3612$ and for Y of $\frac{1}{3}(0.0455 + 0.0455 + 0.1106 = 0.0672$. Since we assumed that X won the East, the expected payoff of winning the

[12] All expected payoffs used here are taken from the calculations contained in Appendix A.

East is 0.3612, and the expected payoff of losing the East is 0.0672. Thus player X obtains an expected payoff of $\left(\frac{X_E}{Z_E}\right)(0.3612) + \left(\frac{Y_E}{Z_E}\right)(0.0672) - X_E$, generating a first order condition $\left(\frac{Y_E}{Z_E^2}\right)(0.3612) - \left(\frac{Y_E}{Z_E^2}\right)(0.0672) = 1$. Because $X_E = Y_E$, due to symmetry, we can solve, giving us $X_E = Y_E = 0.0735$, and thus expected payoffs of 0.1407. Averaging this with the 0.1797 from the cases where North or South is the first round yields a total expected payoff to each player of 0.1602. This is lower than the overall average of all 24 sequential cases where the order is known, which is 0.1641. Since the total benefit to the players is the same, the lower expected payoff means the players must have greater expected expenditures. We summarize this as the following proposition:

Proposition 4. *Contests can be arranged in order of increasing dissipation, with sequential contests with specified order having the lowest expected dissipation, followed by sequential contests with random order and finally simultaneous contests.*

Since players have more difficulty determining which cells are likely to be important in the random case than they do in the pre-specified order case, resource expenditure generally becomes less efficient. This is especially true in the cases where East or West is drawn as the first cell to be contested, because future asymmetries are unknown. When players do not have prior knowledge of the nature of future asymmetries, they cannot exploit these asymmetries. The small differences in expected values in the two cases are driven by the large number of sequences in which the strategies in the random draws end up being the same as in the sequential draw, such as if North or South is drawn in the first round. In the random order case, there is still some information available, unlike the simultaneous case, allowing for more informed decision making and the possibility of an early victory.

Note that when the order of contests is random, any asymmetry in expected payoffs cannot be known until after at least one round has been completed, and winners of previous rounds are known. For example, if we already know that the order will be W-N-E-S in advance, we have an asymmetry which favors player X. However, if all we know is that West is the first round, we may end up with this order, or W-S-E-N, which favors players Y, or W-E-N-S, which favors neither player. Thus what asymmetry, if any, cannot be known until after West has been contested.

The random sequence contest helps explain why the sequential contests have lower resource expenditure in general. As the contest progresses, the relative importance of cell becomes clearer, with some cells becoming irrelevant due to the knowledge of the winners of previously contested cells. Thus the players know what can be safely ignored, and only expend effort in the cells that still matter. In the simultaneous case, the players lack this ability, and thus expend effort on cells that turn out to be irrelevant to the formation of the winning path, such as cases where the winner wins 3 or all 4 cells. This is also why North and South are more valued, as these appear in more winning sets for each player. At least one appears in all winning sets, and North and South forming a winning set by themselves, a fact that is not true for the East and West cells. Thus, lacking other information, North and South are more likely to be relevant than East and West.

6 Discussion

In this section we discuss a number of possible extensions and identify some future research questions. We see that complementarity plays a major role in determining expected payoffs. The differing complementarities between players allows for differing numbers of winning sets to be contested in a given round. These differences in the number of winning sets for each player that have been contested after a given round, create variation in the expected payoffs of the players. Although omitted here in the interest of space, we also see similar results in the intermediate cases, shown in Appendix B, where the structure consists of 2 or 3 rounds. In these cases different expected payoffs for players can occur due to different winning sets. There are other variations of this game possible, some of which we will now discuss.

The most obvious variation would be to increase the size of the grid over which the competition takes place. Although tractability requires the number of cells to be limited, the logic of our results may be of use in structures with more cells. For example, in determining a structure for selling bandwidth on network routers, the owner should take into account the structure of connections the bidders wish to obtain. By identifying the routers that exhibit complementarity, the important routers for the structure may be found, thus allowing a reasonably good, though possibly suboptimal, solution to be found for the seller. We leave this as an open question for future research.

Throughout this paper we have assumed that the structure is externally imposed. If the previous owner of the cells is simply allowed to choose a structure, he will obviously choose one of the structures resulting in a total expected expenditure by the players of 0.75. However, if the structure is the result of decisions made by multiple owners, this may not hold.

If we use the generalized Tullock success function (Tullock, 1980), in which player X has a probability of winning cell i equal to $\frac{X_i^\alpha}{X_i^\alpha+Y_i^\alpha}$, where $0 < \alpha \leq 1$, we can obtain a generalized result for the simultaneous case, where each player spends $\frac{\alpha}{8}$ on the North and South, and $\frac{\alpha}{16}$ on each of the East and West. However, our preliminary investigations show that the sequential cases are quite problematic, as α terms appear as both exponents and as coefficients, giving non-linear behavior with respect to α. Thus, although interesting, the problem of generalized success functions is put aside for future work.

If instead of a costly, unlimited resource, we instead have a costless limited budget, in which unspent resources provide no benefit (Brams and Davis, 1974), we have a slightly different problem. In this case player X attempts to maximize the probability of victory given by

$$P_X\left(\{X_k, Y_k\}_{k\in\{N,S,E,W\}}\middle| R\right) = \sum_{\alpha\in X^\star}\left(\prod_{i\in\alpha, j\in A\setminus\alpha}\frac{X_i}{Z_i}\frac{Y_j}{Z_j}\right)V \tag{4}$$

subject to the budget constraint $X_N + X_S + X_E + X_W = B$.

Although the value of the prize does not matter, in the simultaneous case each player will spend $\frac{1}{3}$ of their budget on the North and South cells, and $\frac{1}{6}$ on the East and West. Therefore, if given a budget $B = \frac{3}{8}V$, the decisions made will be identical to those in the simultaneous case. Calculations are omitted in the interest of space, but proceed in a manner similar to the main case.

However, the sequential cases lead to different solutions. For example, consider the E-N-W-S, E-N-S-W, and E-N-SW cases. There is no advantage to not spending the entire budget, and thus if all four cells are contested, the entire budget will be spent. Players only would reserve resources for future rounds of competition if future rounds are possible. Obviously, once spending decisions for three cells are made, the spending decisions for the fourth cell become set. Thus all three of these structures are equivalent with budget constraints, because after the first two rounds, the only decision left is how to divide the remaining budget between the West and South cells. In contrast, with the costly resource the expected payoffs and expected expenditures vary between these cases. Furthermore, in the costly resource case the identity of the player with the greater expected payoff differs depending on whether there is one of the two four round structures (E-N-W-S or E-N-S-W) listed versus the three round structure (E-N-SW). This obviously cannot be true in the constrained budget case. This difference between the costly resource and costless limited resource cases is likely due to there being no advantage to winning the contest quickly compared to winning it in the final round in the case of budget constraints. While this costless resource case is also an interesting problem, it lies outside the scope of this paper.

References

[1] Vadim V. Anshelevich: The game of Hex: an automatic theorem proving approach to game programming. American Association for Artificial Intelligence Proceedings, 2000

[2] Michael R. Baye, Dan Kovenock, and Casper G de Vries: The incidence of overdissipation in rent-seeking contests. Public Choice *99*, 439-454 (1999)

[3] Elwyn Berlekamp, John Conway, and Richard Guy: Winning Ways For Your Mathematical Plays. Academic Press (1982)

[4] Steven J. Brams and Morton D. Davis: The 3/2's rule in presidential campaigning. The American Political Science Review *68*, 113-134 (1974)

[5] Garikai Campbell: On optimal play in the game of Hex. Integers: Electronic Journal of Combinatorial Number Theory 4, G2 23 pgs. Combinatorial Games Section (2004)

[6] Subhasish M. Chowdhury, Dan Kovenock, and Roman M. Sheremeta: An experimental investigation of Colonel Blotto games. CESifo Working Paper (2009)

[7] Peter Eso and Balazs Szentes: Optimal information disclosure in auctions and the handicap auction. The Review of Economic Studies *74*, 705-731 (2007)

[8] S. Even and R. E. Tarjan: A combinatorial problem which is complete in polynomial space. Journal of the Association for Computing Machinery *23*, 710-719 (1976)

[9] Lawrence Friedman: Game theory models in the allocation of advertising expenditures. Operations Research *6*, 699-709 (1958)

[10] Russell Golman and Scott E. Page: General Blotto: games of allocative strategic mismatch. Public Choice *138*, 279-299 (2009)

[11] Oliver Alfred Gross and R. A. Wagner: A continuous Colonel Blotto game. Rand Corporation Memorandum, RM-408 (1950)

[12] Marcus Hagedorn: The value of information for auctioneers. Journal of Economic Theory *144*, 2197-2208 (2009)

[13] Dan Kovenock and Brian Roberson: Conflicts with multiple battlefields. CESifo Working Paper, 3165 (2010)

[14] Dan Kovenock and Brian Roberson: The optimal defense of networks of targets. Purdue Working Paper 1251 (2010)

[15] Dmitriy Kvasov: Contests with limited resources. Journal of Economic Theory *136*, 738-748 (2007)

[16] Jean-Francois Laslier: How two-party competition treats minorities. Review of Economic Design *7*, 297-307 (2002)

[17] Jean-Francois Laslier and Nathalie Picard: Distributive politics and electoral competition. Journal of Economic Theory *103*, 106-130 (2002)

[18] Assar Lindbeck and Jöregen W. Weibull: Balanced budget redistribution as the outcome of political competition. Public Choice *52*, 273-297 (1987)

[19] Brian Roberson: The Colonel Blotto game. Economic Theory *29*, 1-24 (2006)

[20] Alexander R.W. Robson: Multi-item contests. Australian National University Working Paper, 446 (2005)

[21] James M Snyder: Election goals and the allocation of campaign resources. Econometrica *57*, 637-660 (1989)

[22] Gordon Tullock: On the efficient organization of trials. Kyklos *28*, 745-762 (1975)

[23] Gordon Tullock: Efficient rent seeking. Towards a Theory of the Rent Seeking Society, edited by Gordon Tullock, Robert D. Tollison, James M. Buchanan, Texas A & M Press, 3-15 (1980)

[24] Eric W. Weisstein: Game of Hex. mathworld.wolfram.com (Accessed March 2010)

7 Appendix

7.1 Appendix A: Polar Cases: Simultaneous versus Sequential Cases

For Proposition 1 we will start by computing the expected payoffs from the simultaneous and sequential cases. For the simultaneous case, we first prove a lemma which shows that in any Nash equilibrium, the players will play a symmetric strategy.

Case 1: All Four Cells Simultaneously

Lemma 1 *In any Nash equilibrium of the simultaneous contest game, neither player will play an asymmetric strategy.*

Proof:

This proof is broken into three parts – one where only one player uses an asymmetric strategy and two cases where both use asymmetric strategies. Recall that symmetry here refers to players expending identical effort on North and South, as well as on East and West. In each case we will show that a player with an asymmetric strategy between North and South (or East and West) can deviate to a strategy with a higher probability of winning both cells and a lower probability of losing both cells while maintaining the same total expenditures. As the only relevant distinctions are winning zero, one, or two of these two cells, showing the existence of a strategy which increases the probability of winning both cells while decreasing the probability of losing both cells without any change in expenditure is enough to demonstrate the original strategy is not Nash, which we demonstrate first, before moving to the three different asymmetries.[13]

Consider a Case J where the probability of winning both cells increases by P_2 while the probability of losing both decreases by $P_0 < P_2$. Thus the probability of winning exactly one cell P_1 must decrease by $P_2 - P_0$, as the sum of the changes $P_0 + P_1 + P_2$ must equal zero, as the sum of all probabilities must be 1. Now consider Case K, where the probability of winning two cells increases by P_0 while the probability of losing both cells decreases by P_0, leaving the probability of winning exactly one cell unchanged. Case K has a greater expected value than the starting case, as it consists solely of increasing the probability of winning both cells and decreasing the probability of losing both cells, a clear improvement. However, increasing the probability of winning both cells by the remaining $P_2 - P_0$ and decreasing the probability of winning exactly one cell by $P_2 - P_0$ while leaving the probability of winning zero cells the same is also certainly an improvement. Making

[13]We would like to thank an anonymous referee for raising this point.

these changes to Case K gives us Case J. Thus, by transitivity, the expected value must be greater for the new probabilities than the original probabilities, and thus the original strategy could not be a Nash equilibrium.

Now we can move into the three different parts of the proof.

Part A: Without loss of generality, assume that we have a Nash equilibrium with player X is playing an asymmetric strategy. Assume player X spends $a < b$ on the North and b on the South, while player Y who is playing the symmetric strategy spends c on each. Recall that players are indifferent between North and South *a priori*, as well as between East and West. Player X will have a probability of winning both North and South of

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+c}\right) = \frac{ab}{ab+ac+bc+c^2} \tag{5}$$

However, if player X deviates to a strategy of $a+\epsilon$ in the North and $b-\epsilon$ in the South for small $\epsilon > 0$, thus keeping total expenditures on the two cells constant, the probability of winning both becomes

$$\left(\frac{a+\epsilon}{a+\epsilon+c}\right)\left(\frac{b-\epsilon}{b-\epsilon+c}\right) = \frac{ab+b\epsilon-a\epsilon-\epsilon^2}{ab+ac+bc+c^2+b\epsilon-a\epsilon-\epsilon^2} \tag{6}$$

which is greater than the value in equation (5) as we are adding the same positive value to both the numerator and denominator of a positive fraction, making the new fraction closer to 1. Similarly, the probability of losing both cells decreases from

$$\left(\frac{c}{a+c}\right)\left(\frac{c}{b+c}\right) = \frac{c^2}{ab+ac+bc+c^2} \Rightarrow \left(\frac{c}{a+\epsilon+c}\right)\left(\frac{c}{b-\epsilon+c}\right) = \frac{c^2}{ab+ac+bc+c^2+b\epsilon-a\epsilon-\epsilon^2}$$

which must be smaller as the denominator has been increased with a constant numerator. As the probability of winning both cells has increased, and the probability of losing both cells has decreased, the original strategy could not have been a Nash equilibrium. If $a > b$, we simply reverse the direction of deviation, moving expenditure from a towards b.

Part B1: We assume player X spends $a < b$ on the North and b on the South, while player Y also has an asymmetric strategy where Y expends c in the North and d in the South. Further we assume $(d-c) < (b-a)$. [14] Thus we obtain a probability of winning both cells of

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+d}\right) = \frac{ab}{ab+ac+bd+cd} \tag{7}$$

and if player X deviates to a strategy of $a+\epsilon$ in the North and $b-\epsilon$ in the South, the probability

[14] If we reverse this inequality, the equivalent results hold if we switch players X and Y.

of winning both becomes

$$\left(\frac{a+\epsilon}{a+\epsilon+c}\right)\left(\frac{b-\epsilon}{b-\epsilon+d}\right)=\frac{ab+b\epsilon-a\epsilon-\epsilon^2}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2} \tag{8}$$

which again yields a greater probability of winning both cells. Similarly, the probability of losing both cells decreases from

$$\left(\frac{c}{a+c}\right)\left(\frac{d}{b+d}\right)=\frac{cd}{ab+ac+bd+cd}\Rightarrow\left(\frac{c}{a+\epsilon+c}\right)\left(\frac{d}{b-\epsilon+d}\right)=\frac{cd}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2}$$

so again, the original strategy could not have been a Nash equilibrium. Again, we arbitrarily chose $a<b,(d-c)<(b-a)$, if $a>b,(d-c)<(b-a)$ we would instead subtract ϵ from a and add it to b.

Part B2: We assume player X spends $a<b$ on the North and b on the South, while player Y also has an asymmetric strategy where Y expends c in the North and d in the South, with $(d-c)<(b-a)$. We must break into two cases, depending on the relationship between $\frac{d-c}{b-a}$ and $\frac{cd}{ab}$.

Part B2(i): $\frac{d-c}{b-a}\leq\frac{cd}{ab}$

The probability of X winning both cells is

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+d}\right)=\frac{ab}{ab+ac+bd+cd} \tag{9}$$

and if player X deviates to a strategy of $a+\epsilon$ in the North and $b-\epsilon$ in the South, the probability of winning both becomes

$$\left(\frac{a+\epsilon}{a+\epsilon+c}\right)\left(\frac{b-\epsilon}{b-\epsilon+d}\right)=\frac{ab+b\epsilon-a\epsilon-\epsilon^2}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2} \tag{10}$$

which yields a greater probability of winning both cells. Similarly, the probability of losing both cells decreases from

$$\left(\frac{c}{a+c}\right)\left(\frac{d}{b+d}\right)=\frac{cd}{ab+ac+bd+cd}\Rightarrow\left(\frac{c}{a+\epsilon+c}\right)\left(\frac{d}{b-\epsilon+d}\right)=\frac{cd}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2}$$

so the original strategy could not have been a Nash equilibrium.

Part B2(ii): $\frac{d-c}{b-a}>\frac{cd}{ab}$

First, note that $d>c$, as we know $a,b,c,d,$ and $(b-a)$ are all positive. We begin with a probability of Y winning both cells of

$$\left(\frac{c}{a+c}\right)\left(\frac{d}{b+d}\right)=\frac{cd}{ab+ac+bd+cd} \tag{11}$$

and if player Y deviates to a strategy of $c+\epsilon$ in the North and $d-\epsilon$ in the South, the probability of winning both becomes

$$\left(\frac{c+\epsilon}{c+\epsilon+a}\right)\left(\frac{d-\epsilon}{d-\epsilon+b}\right)=\frac{cd-c\epsilon+d\epsilon-\epsilon^2}{ab+ac+bd+cd-a\epsilon+b\epsilon-c\epsilon+d\epsilon-\epsilon^2} \tag{12}$$

which again yields a greater probability of winning both cells. Similarly, the probability of losing both cells decreases from

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+d}\right)=\frac{ab}{ab+ac+bd+cd}\Rightarrow\left(\frac{a}{c+\epsilon+a}\right)\left(\frac{b}{d-\epsilon+b}\right)=\frac{ab}{ab+ac+bd+cd-a\epsilon+b\epsilon-c\epsilon+d\epsilon-\epsilon^2}$$

so again, the original strategy could not have been a Nash equilibrium. If $b<a$, we would have $d<c$, and again would switch the direction of the deviation. Thus though player X may not gain from deviating in the case, player Y will, and thus the original strategy could not have been Nash.

Thus at least one player will always benefit from deviating if a player is playing an asymmetric strategy, therefore the original strategies could not have been Nash. Thus we have eliminated asymmetric strategies from consideration, and need only consider symmetric strategies. □

Computing payoffs for the simultaneous case:

Now we can solve the optimization problem and thus obtain the Nash equilibrium. Without loss of generality, we will solve X's optimization problem, which is equivalent to maximizing equation 2. Moreover, due to Lemma 1 $X_N=X_S$ and $X_E=X_W$. Hence, equation 2 can be written as:

$$\left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right)^2+2\left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_E}{Z_E}\right)+\left(\frac{X_N}{Z_N}\right)^2\left(\frac{Y_E}{Z_E}\right)-2X_N-2X_E. \tag{13}$$

This gives us the first order conditions with respect to X_N and X_E of:

$$\frac{\partial U_X}{\partial X_N}=\left(\frac{Y_N}{Z_N^2}\right)\left(\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_S}{Z_S}\right)+\left(\frac{Y_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right)\right)-1=0$$
$$\frac{\partial U_X}{\partial X_E}=\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)-1=0$$

As the equations for this case are symmetric for East versus West, North versus South and because of this, X versus Y, these are the only equations required. Making this substitution and simplifying gives us the system $\frac{1}{4X_N}\left(\frac{1}{4}+\frac{1}{4}\right)=1$ and $\frac{1}{4X_E}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=1$. Solving this system of equations, we obtain $X_N=\frac{1}{8}; X_E=\frac{1}{16}$.

Second order conditions are omitted, however intuitively as we have only one solution for the first order conditions, equation 2 is continuous, and if a player expends ϵ arbitrarily close to zero on each cell they have an expected payoff arbitrarily close to zero, while expending greater than 1 on each cell leads to a negative expected payoff, the solution must be a maximum.

Case 2 : Four cells sequentially

We now will solve the sequential cases. Combining these with the simultaneous cases will give us Proposition 1, while comparing the sequential cases will prove Proposition 2. In order to obtain these results, we must work backwards from the terminal nodes. If only one cell remains to be contested, either the cell is irrelevant as there is already a winner, or the winner of this one cell will win the contest. Taking the remaining cell to be East without loss of generality, we have $U_X(., R_4) = \left(\frac{X_E}{Z_E}\right) - X_E$, and $U_Y(., R_4) = \left(\frac{Y_E}{Z_E}\right) - Y_E$. Taking derivatives gives us $\left(\frac{X_E}{Z_E^2}\right) = 1, \left(\frac{Y_E}{Z_E^2}\right) = 1$, so $X_E = Y_E = \frac{1}{4}$, giving $U_X(., R_4) = U_Y(., R_4) = \frac{1}{4}$.

Now we can work backwards to the previous stage. If there are two cells remaining, there are three possibilities. If the contest has already been won, they are both irrelevant, and we are done. However, if only one of the two remaining cells is relevant, we ignore the irrelevant cell, and this reduces to the above. Finally, there is the possibility that both are relevant, with one player needing to win both and the other needing to win either. We will assume without loss of generality that North and East remain to be contested sequentially, with Player X requiring both to win. Then $U_X(., R_3, R_4) = \left(\frac{X_N}{Z_N}\right)\frac{1}{4} - X_N$, $U_Y(., R_3, R_4) = \left(\frac{Y_N}{Z_N}\right) + \left(\frac{X_N}{Z_N}\right)\frac{1}{4} - Y_N$ gives the expected utility for each player in the North round. Taking derivatives gives us

$$\left(\frac{Y_N}{Z_N^2}\right)\frac{1}{4} = 1$$
$$\left(\frac{X_N}{Z_N^2}\right) - \left(\frac{X_N}{Z_N^2}\right)\frac{1}{4} = \left(\frac{X_N}{Z_N^2}\right)\frac{3}{4} = 1$$

Solving this gives $X_N = \frac{3}{64}$, while $Y_N = \frac{9}{64}$, and thus expected payoffs $U_X(., R_3, R_4) = \frac{1}{64}, U_Y(., R_3, R_4) = \frac{43}{64}$. These two solutions will be used extensively in the subcases below. However we cannot obtain a general set of solutions for the second round without having the specifics of the complementarity, so this is reserved for the subcases.

Subcase 2a: *North or South as the first round*

After the first round, one of the remaining cells will become irrelevant, with the winner of the first round needing to win one of the remaining two cells, and the loser of the first round needing to win both. Thus, if North is the first cell, we have $U_X = \left(\frac{X_N}{Z_N}\right)\frac{43}{64} + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$. Taking the derivative with respect to X_N gives $\left(\frac{X_N}{Z_N^2}\right)\frac{43}{64} - \left(\frac{X_N}{Z_N^2}\right)\frac{1}{64} = 1$. Due to symmetry $X_N = Y_N$, so solving gives us $X_N = Y_N = \frac{21}{128}$ and $U_X = U_Y = \frac{23}{128}$.

Subcase 2b: *East and West in the first two rounds*

Without loss of generality, we consider the cases where East is the first round, results for West as opening round are symmetric. If player X wins the East, winning the West means he will need either the North or South, while losing the West means that the South is irrelevant, and the North will determine the overall winner. These have expected payoffs of $\frac{43}{64}$ and $\frac{1}{4}$ respectively for player X and $\frac{1}{64}$ and $\frac{1}{4}$ for player Y. Thus, if player X wins the East, the expected payoff functions for the second round are

$$U_X(., R_2, R_3, R_4) = \left(\frac{X_W}{Z_W}\right)\frac{43}{64} + \left(\frac{Y_W}{Z_W}\right)\frac{1}{4} - X_W$$
$$U_Y(., R_2, R_3, R_4) = \left(\frac{Y_W}{Z_W}\right)\frac{1}{4} + \left(\frac{X_W}{Z_W}\right)\frac{1}{64} - Y_W$$

This gives us first order conditions of

$$\left(\frac{Y_W}{Z_W^2}\right)\frac{27}{64} = \left(\frac{X_W}{Z_W^2}\right)\frac{15}{64} = 1$$

and so $15X_W = 27Y_W$. Thus $X_W = \frac{27}{15}Y_W$, which when placed into the first order conditions gives us $X_W = \frac{(15)(27^2)}{(42^2)(64)}$, $Y_W = \frac{(15^2)(27)}{(42^2)(64)}$. Using these payoffs in the expected payoff functions gives $U_X = \frac{47907}{112896}$ and $U_Y = \frac{5139}{112896}$. These payoffs will be reversed if player Y wins the East.

Thus, in the initial round, the expected payoff function for player X is $\left(\frac{X_E}{Z_E}\right)\frac{47907}{112896} + \left(\frac{Y_E}{Z_E}\right)\frac{5139}{112896} - X_E$, giving a first order condition of $\left(\frac{Y_E}{Z_E^2}\right)\frac{42768}{112896} = 1$, with $X_E = Y_E$ due to symmetry. Thus, the optimal strategy is $X_E = Y_E = \frac{42768}{451584}$, which yields expected payoffs of $\frac{63486}{451584}$.

Subcase 2c: *East or West in Round One and North or South in Round Two*

Without loss of generality, we consider the cases where East is the first round. If player X wins the East, winning the North means victory, while losing North means he must win both West and South, for an expected payoff of $\frac{1}{64}$ for X and $\frac{43}{64}$ for Y. Thus the expected payoff functions in the second round if X won the East are

$$U_X(., R_2, R_3, R_4) = \left(\frac{X_N}{Z_N}\right) + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$$
$$U_Y(., R_2, R_3, R_4) = \left(\frac{Y_N}{Z_N}\right)\frac{43}{64} - Y_N$$

This gives the first order conditions of

$$\left(\frac{Y_N}{Z_N^2}\right)\frac{63}{64} = \left(\frac{X_N}{Z_N^2}\right)\frac{43}{64} = 1$$

Thus we have $43X_N = 63Y_N$, which with our first order condition gives $X_N = \frac{(43)(63^2)}{(64)(106^2)}$ and $Y_N = \frac{(43^2)(63)}{(64)(106^2)}$. Plugging these into the expected payoff equations give approximations of $U_X \approx 0.2373$, $U_Y \approx 0.1106$.

If player Y wins the East, winning either the South or North and West wins. This is the same as in the E-W-N-S case, which gives decimal approximations of $U_X \approx 0.0455$ and $U_Y \approx 0.4245$.

Thus for the first round, we have expected payoff functions of

$$U_X = \left(\frac{X_E}{Z_E}\right).2373 + \left(\frac{Y_E}{Z_E}\right).0456 - X_E$$
$$U_Y = \left(\frac{Y_E}{Z_E}\right).4245 + \left(\frac{X_E}{Z_E}\right).1106 - Y_E$$

These yield first order conditions of

$$\left(\frac{Y_E}{Z_E^2}\right)(.2373 - .0455) = \left(\frac{X_E}{Z_E^2}\right)(.4245 - .1106) = 1$$
$$\left(\frac{Y_E}{Z_E^2}\right).1918 = \left(\frac{X_E}{Z_E^2}\right).3139 = 1$$
$$Y_E.1918 = X_E.3139$$
$$Y_E \approx X_E 1.6365$$

Thus we find $X_E \approx 0.0452$, $Y_E \approx 0.0739$. Placing these in our expected payoff function gives $U_X \approx 0.0731$, $U_Y \approx 0.2315$.

Combining cases 1 and 2 gives us Proposition 1, while comparing the expected payoffs for each player across case 2 gives us Proposition 2. □

7.2 Appendix B: The Other Sequential Contests

To explain the other types of sequential contests here we will go through the details of one case. The other cases are similar and details of these cases can be found in the working paper version. [15]

The case solved will be that of two rounds of selling, each round containing two cells. This case will require a solution to the expected payoffs of each player if only 2 cells remain, they are being auctioned simultaneously, and one player requires both cells for victory, while the other player requires either for victory. Without loss of generality, we will take player X as needing both the North and East cells. If player X needs both cells, his expected payoff is $\left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) - X_N - X_E$, while player Y has an expected payoff of $1 - \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) - Y_N - Y_E$. Taking derivatives, we obtain the following set of first order conditions

$$\left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) = \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_E}{Z_E^2}\right) = \left(\frac{X_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E^2}\right) = 1 \quad (14)$$

After a bit of algebra, we obtain that $X_N = Y_N = X_E = Y_E = \frac{1}{8}$, so player X has an expected payoff of 0, while player Y has an expected payoff of $\frac{1}{2}$.

[15] http://xythos.lsu.edu/users/mwiser1/Hex

We now have enough information to write the expected payoff equations and thus solve this case. However, we will need to break this case into three subcases. The results for the other cases are summarized in the final table.

Subcase 3a: East and West as the First Round If player X wins both the East and West in the initial round, they will complete a winning set with either North or South, and thus X will have an expected payoff of $\frac{1}{2}$, while Y will have an expected payoff of 0. If players X and Y split East and West, only one of North and South will matter, and thus each player has an expected payoff of $\frac{1}{4}$. Thus, player X has an expected payoff of

$$\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2}+\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4}+\left(\frac{Y_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4}-X_E-X_W$$

As East and West are symmetric, as well as players X and Y, we know that $X_E = X_W = Y_E = Y_W$. Thus, taking the derivative with respect to X_E gives us the following

$$\begin{gathered}\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2}+\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4}-\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4}-1\\ =\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_E}{Z_E}\right)\frac{1}{2}-1=\left(\frac{Y_E}{Z_E^2}\right)\frac{1}{4}-1=\frac{1}{4X_E}\frac{1}{4}-1\end{gathered}$$

Setting this equal to zero gives us first order conditions of $X_E = X_W = Y_E = Y_W = \frac{1}{16}$, and using this gives us expected payoffs for each player of $\frac{1}{8}$.

Type	**Order**	$E[U_X]$	$E[U_Y]$	EV[A]
4	NESW	.125	.125	.75
3-1	NSW-E, NSE-W	.13	.13	.74
3-1	EWN-S, EWS-N	.1563	.1563	.6875
2-2	NE-WS, WS-NE	.3396	.1793	.4811
2-2	SE-NW, NW-SE	.1793	.3396	.4811
2-2	NS-EW	.125	.125	.75
2-2	EW-NS	.125	.125	.75
1-3	N-ESW, S-ENW	.125	.125	.75
1-3	E-NSW, W-NSE	.2292	.2292	.5417
2-1-1	EW-N-S, EW-S-N	.1328	.1328	.7344
2-1-1	NE-S-W, SW-N-E, NE-W-S, SW-E-N	.1583	.1471	.6947
2-1-1	NW-S-E, SE-N-W, NW-E-S, SE-W-N	.1471	.1583	.6947
2-1-1	NS-E-W, NS-W-E	.125	.125	.75
1-2-1	N-EW-S, S-EW-N	.1797	.1797	.6719
1-2-1	N-ES-W, S-WN-E	.1006	.2269	.6725
1-2-1	S-EN-W, N-WS-E	.2269	.1006	.6725
1-2-1	E-NS-W, W-NS-E	.2368	.2368	.5265
1-2-1	E-NW-S, W-SE-N	.2040	.3372	.4588
1-2-1	W-NE-S, E-SW-N	.3372	.2040	.4588
1-1-2	N-S-EW, S-N-EW	.1797	.1797	.6719
1-1-2	N-E-SW, S-W-NE	.1006	.2269	.6725
1-1-2	N-W-SE, S-E-NW	.2269	.1006	.6725
1-1-2	E-S-NW, W-N-SE	.0933	.2003	.7064
1-1-2	E-N-SW, W-S-NE	.2003	.0933	.7064
1-1-2	E-W-NS, W-E-NS	.125	.125	.75
1-1-1-1	N-(E,W,S), S-(E,W,N)	.1797	.1797	.6719
1-1-1-1	E-W-N-S, E-W-S-N, W-E-N-S, W-E-S-N	.1406	.1406	.7188
1-1-1-1	E-N-W-S, W-S-E-N, E-N-S-W, W-S-N-E	.0731	.2315	.6954
1-1-1-1	W-N-E-S, E-S-W-N, W-N-S-E, E-S-N-W	.2315	.0731	.6954

Table 2: Expected Payoffs For Each Player and Seller Under All Structures