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Efficient networks for a class of games with global spillovers

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Abstract

In this paper we characterize efficient networks for network formation games with global spillovers, that satisfy convexity and sub-modularity properties. This allows us to complete the work of Goyal and Joshi (2006) and Westbrock on collaborative oligopoly networks. In particular, we establish that efficient networks are nested split graphs in this class of games.

JEL code: Classification: C70, D85.
Key Words: networks, efficiency, convexity, sub-modularity.

1 Introduction

Research shows that collaborations among firms in innovative activities has become widespread, especially in industries characterized by rapid technological change (*e.g.* the pharmaceutical, chemical and computer industries, see Hagedoorn, 2002; Powell et al., 2005). R&D collaborations allow firms to improve their ability to innovate. In addition, they provide access to indirect spillovers since they allow the diffusion of information across firms (see Ahuja, 2000; Powell et al., 2005). The increasing importance of this phenomenon has also spurred economic research on the structural features of the network of R&D collaborations, and on their impact on industry performance. Empirical studies have shown that such real-world networks are typically asymmetric. More precisely, there exist simultaneously firms having many collaborations and others with few links (see for example Powell et al., 2005).

R&D networks have been theoretically studied by Goyal and Moraga-Gonzalez (2001), Goyal and Joshi (2003, GJ), Westbrock (2010). GJ (2003), focus on the trade-off between the benefits from collaborations and the costs of maintaining them. More precisely, the authors propose a game of network formation in a homogeneous-product oligopoly. In the first stage of the game, firms form *bilateral collaborative* links. These links allow them to decrease their marginal costs. In the second stage, all firms compete in the product market. GJ (2003) examine stable networks, that is networks where no pair of unlinked firms has an incentive to form a link, and no firm has an incentive to remove one of its link unilaterally. They find that stable networks contain a dominant group (a subset of firms connected together) and other firms are isolated. Westbrock (2010) examines efficient networks, that is networks which maximize the social welfare in this context. He also studies the relationship between stable networks and efficient networks. He finds that non-complete and non-empty efficient networks are asymmetric. More precisely, he shows that non-empty efficient networks are interlinked stars (Proposition 1, p. 358, 2010),[1] that is networks where the firms which are involved in the largest number of links are connected with each non isolated firm. Moreover, he establishes conditions under which stable and efficient networks coincide.

Further, GJ (2006) have established that their basic model (GJ, 2003) belongs to a larger class of network formation games: the class of games with global spillovers satisfying convexity and sub-modularity.[2] They establish that the architecture of stable networks in their basic model (2003) is also the architecture of stable networks in the class of network formation games with global spillovers which satisfy convexity and sub-modularity.

Networks efficiency is a major performance criterion for network designers or planers, and play a prominent role in the traditional network literature. Consequently, an important task consists in characterizing efficient networks in a general class of games of network formation which satisfies convexity and sub-modularity properties. Apart from Westbrock, another im-

[1]Westbrock shows that an efficient network has either a dominant group architecture, or an interlinked star architecture (Proposition 1, p. 358, 2010). In this paper, we modify the definition of interlinked stars slightly: with our definition networks that have a dominant group architecture have also an interlinked star architecture.

[2]Sub-modularity property is called strategic substitute property in Goyal and Joshi (2006).

portant paper in the domain of networks is the work by Haller (2013). He characterizes efficient networks in the context of infrastructure building under private-public partnerships.

In this paper, we focus on *efficient* networks where the total payoffs is maximized and *global efficient networks* where the total payoff function of the players plus another function is maximized. This additional function could be used to capture payoffs of those who are not players in the game itself. For example in the context of oligopolies this additional function could be used to capture consumer surplus.

Our paper highlights a class of networks called nested split graphs (NSG) for describing efficient networks. In these networks, neighborhood of players[3] are nested. More precisely, if a player i has formed more links than player j, then all the neighbors of j are in the neighborhood of i. Three other papers focus on NSG. First, König et al. (2012) investigate R&D collaborations in a model with spillovers along network lines. Second, König et al. (2011) explore in details the topological properties of NSG, which they find to be stable graphs of a dynamic, non-cooperative network formation game. Third, Belhaj et al. (2013) analyse network design issues instead of strategic incentives for link formation. In the current paper, we establish the relationship between NSG and efficient networks for the class of network formation games with global spillovers which satisfy convexity and sub-modularity.

Our paper adds to the literature in several ways. First, it completes GJ (2006) since we identify efficient networks in network formation games with the convexity and sub-modularity properties. A caveat is needed here - we require an additional convexity condition on the payoffs of the players: the payoffs of each player is convex with the number of links in which she is not involved.[4] Moreover, the Cournot oligopoly and cost reduction game defined by GJ (2003) belongs to the class of games defined in the present paper. Second, we extend Westbrock (2010) in the following directions.

1. First, we refine his results concerning the architecture of efficient networks by relating the architecture used in this paper (NSG) to the architecture found by Westbrock (interlinked

[3]The neighbors of player i are the players with whom player i is linked.

[4]This assumption allows us to take into account the impact of an additional link on the payoffs of players who are not involved in.

stars).[5] Moreover, we provide a simple method allowing us to check whether a network is a candidate for being an efficient network.

2. Second, we do not use the same approach as the one used by Westbrock. Indeed, Westbrock provides general conditions on the total welfare function[6] to obtain his results while we provide general conditions on the individual payoff functions. This makes our results interesting for comparison purposes since stable networks are typically identified using conditions on individual payoffs.

This paper is organized as follows. In section 2, we introduce the model and the basic properties of the payoff function and we show that the basic model of GJ (2003) satisfies these properties. We also define the architectures used in this paper. In section 3, we establish our main result: efficient networks are NSG.

2 Model setup

2.1 Networks

To simplify notation, we use $[\![a, b]\!] = \{a, a + 1, \ldots, b\}$ for $a, b \in \mathbb{N}$ and denote by $|Y|$ the number of elements in the set Y. Let $N = [\![1, n]\!]$ be the set of players, with i and j as typical members of N. For any $i, j \in N$, the pair-wise relationship between the two players is captured by a binary variable, $g_{i,j} \in \{0, 1\}$; $g_{i,j} = 1$ means that a link, ij, exists between players i and j, $g_{i,j} = 0$ means that there is no link between i and j. A network $\boldsymbol{g} = \{(g_{i,j})_{i\in N, j\in N}\}$ is a formal description of the links that exist between the players. Let $\mathcal{G}$ denote the set of all simple networks when there are n players, that is networks without loops (a player i cannot form a link with herself) or multiple links (players i and j can establish at most one link between them). We denote by $\bar{\boldsymbol{g}}$ the complement of network $\boldsymbol{g}$. We have $\bar{g}_{i,j} = 1 - g_{i,j}$ for all $i \in N$ and $j \in N \setminus \{i\}$.

Let $X \subset N$ be a set of players. We define $\boldsymbol{g}^{-X}$ as a network which is identical to $\boldsymbol{g}$ except that players in X and all their links have been removed. Let $g(i) = \{(g_{i,j})_{j\in N\setminus\{i\}} : g_{i,j} = 1\}$ be the set of links in which player i is involved in $\boldsymbol{g}$, and let $|g(i)|$ be the number of

[5]Interlinked stars are not always NSG.

[6]Westbrock (2010) defines a specific convexity property on the aggregate social welfare function.

links in which i is involved in $\boldsymbol{g}$, which is called the degree of player i. If $|g(i)| = 0$, we say that $i \in N$ is an isolated player in $\boldsymbol{g}$. Similarly, we define $g^{-\{i\}}(j)$ as the set of links in which j is involved in $\boldsymbol{g}^{-\{i\}}$. Let $L(\boldsymbol{g}) = (1/2) \sum_{i \in N} |g(i)|$ be the number of links in $\boldsymbol{g}$. Hence $L\left(\boldsymbol{g}^{-\{i\}}\right) = (1/2) \sum_{j \in N \setminus \{i\}} |g^{-\{i\}}(j)|$ is the number of links in $\boldsymbol{g}^{-\{i\}}$. We define $N_i(\boldsymbol{g}) = \{j \in N \setminus \{i\} : g_{i,j} = 1\}$ as the set of players with whom player i is linked in $\boldsymbol{g}$. Let $\boldsymbol{g} + ij$ denote the network obtained by replacing $g_{i,j} = 0$ in network $\boldsymbol{g}$ by $g_{i,j} = 1$. Similarly, let $\boldsymbol{g} - ij$ denote the network obtained by replacing $g_{i,j} = 1$ in network $\boldsymbol{g}$ by $g_{i,j} = 0$. We set $\mathcal{E}^{\boldsymbol{g}} = \{|g(i)| \neq 0 : i \in N\}$ and $\mathcal{E}^{\boldsymbol{g}}(k) = \{\ell \in \mathcal{E}^{\boldsymbol{g}} : \ell < k\}$. Further define $D^{\boldsymbol{g}}_\ell = \{i \in N : |\mathcal{E}^{\boldsymbol{g}}(|g(i)|)| + 1 = \ell\}$. We denote by $D^{\boldsymbol{g}}_0$ the set of isolated players (this set can be empty). Then the vector $\boldsymbol{D}^{\boldsymbol{g}} = (D^{\boldsymbol{g}}_0, D^{\boldsymbol{g}}_1, \ldots, D^{\boldsymbol{g}}_m)$, with $D^{\boldsymbol{g}}_m = D^{\boldsymbol{g}}_{|\mathcal{E}^{\boldsymbol{g}}|}$, is called the degree partition of $\boldsymbol{g}$. In network $\boldsymbol{g}$ drawn in Figure 1, we have $\mathcal{E}^{\boldsymbol{g}} = \{2, 3, 5\}$. Moreover, we have $\mathcal{E}^{\boldsymbol{g}}(2) = \emptyset$, $\mathcal{E}^{\boldsymbol{g}}(3) = \{2\}$, and $\mathcal{E}^{\boldsymbol{g}}(5) = \{2, 3\}$. We have for instance $D^{\boldsymbol{g}}_2 = \{c, d, h\}$ since players c, d, h are players who satisfy $|g(c)| = |g(d)| = |g(h)| = 3$ and $|\mathcal{E}^{\boldsymbol{g}}(3)| + 1 = 2$. Moreover, we have $D^{\boldsymbol{g}}_0 = \{j\}$, $D^{\boldsymbol{g}}_1 = \{a, b, e, g, i\}$, $D^{\boldsymbol{g}}_2 = \{c, d, h\}$, and $D^{\boldsymbol{g}}_3 = \{f\}$.

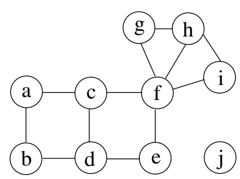


Figure 1: Network $\boldsymbol{g}$

We now define the main network configurations that are extensively used in our model. An *empty network* is a network where all players are isolated. Westbrock (2010) defines a class of networks which plays a crucial role in our analysis: the class of *interlinked stars*. We modify the definition of this class of networks in order to simplify the presentation.[7]

Definition 1 *In an interlinked star $\boldsymbol{g}$, each player $i \in D^{\boldsymbol{g}}_m$ is linked with each player $j \in \bigcup_{\ell=1}^{m} D^{\boldsymbol{g}}_\ell \setminus \{i\}$.*

[7]In the original definition given by Westbrock (2010), an interlinked star $\boldsymbol{g}$ contains at least two groups of players with strictly positive degree. A network where there is only one group of players with strictly positive degree is called by Westbrock a network with *dominant group architecture* (see network $\boldsymbol{g}$ in Figure 2). To sum up with our definition interlinked stars are both the interlinked stars of Westbrock and the networks which have the dominant group architecture.

We now define the key architecture of this paper: the *nested split graphs* (NSG).

Definition 2 (Mahadev and Peled, Theorem 1.2.4, pg. 10, 1995). *Consider a nested split graph $\boldsymbol{g}$. For each player $i \in D_\ell^{\boldsymbol{g}}$, $\ell = 1, \dots, m$,*

$$N_i(\boldsymbol{g}) = \begin{cases} \bigcup_{j=1}^{\ell} D_{m+1-j}^{\boldsymbol{g}}, & \text{if } \ell = 1, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\ \\ \bigcup_{j=1}^{\ell} D_{m+1-j}^{\boldsymbol{g}} \setminus \{i\}, & \text{if } \ell = \left\lfloor \frac{m}{2} \right\rfloor + 1, \dots, m. \end{cases}$$

However, in this paper, we will use an alternative definition of NSG that allows us to relate the notions of interlinked star and NSG.

Definition 3 *Let $\boldsymbol{g}$ be such that $|g(1)| \geq |g(2)| \geq \ldots \geq |g(n)|$. Network $\boldsymbol{g}$ is a NSG if $\boldsymbol{g}$ is an interlinked star, and there exists $\ell \in [\![1, n-1]\!]$ such that $\boldsymbol{g}^{-[\![1,\ell']\!]}$ is an interlinked star for each $\ell' \in [\![1, \ell]\!]$ while $\boldsymbol{g}^{-[\![1,\ell'']\!]}$ is an empty network for each $\ell'' \in [\![\ell+1, n-1]\!]$.*

We now establish that Definitions 2 and 3 are equivalent. It is easy to see that Definition 2 implies Definition 3. We show that Definition 3 implies Definition 2. First, in Definition 3, players in $D_m^{\boldsymbol{g}}$ are linked with all players in $\cup_{\ell=1}^{m} D_\ell^{\boldsymbol{g}}$. Next, with Definition 3 players in $D_{m-1}^{\boldsymbol{g}}$ are linked with all players in $\cup_{\ell=2}^{m-1} D_\ell^{\boldsymbol{g}}$. Indeed, suppose that there exists $D_k^{\boldsymbol{g}}$, $k \neq 1$, such that players in this set are linked with players in $D_m^{\boldsymbol{g}}$, but not linked with players in $D_{m-1}^{\boldsymbol{g}}$. Then, we obtain a contradiction since $\boldsymbol{g}^{-D_m^{\boldsymbol{g}}}$ is not an interlinked star. We can reiterate this argument for each $D_\ell^{\boldsymbol{g}}$, with $\ell = \left\lfloor \frac{m}{2} \right\rfloor + 1, \dots, m$. We have for $i \in D_\ell^{\boldsymbol{g}}$, with $\ell = \left\lfloor \frac{m}{2} \right\rfloor + 1, \dots, m$, $N_i(\boldsymbol{g}) = \bigcup_{j=1}^{\ell} D_{m+1-j} \setminus \{i\}$. Similarly, we have for $i \in D_\ell^{\boldsymbol{g}}$, with $\ell = 1, \dots, \left\lfloor \frac{m}{2} \right\rfloor$, $N_i(\boldsymbol{g}) = \bigcup_{j=1}^{\ell} D_{m+1-j}$.
It is worth noting that Definition 3 implies that a NSG is an interlinked star.
We now illustrate the NSG architecture through an example. In Figure 2, network $\boldsymbol{g}$ is such that $|g(i)| \geq |g(i+1)|$ for each $i \in [\![1, 6]\!]$. Network $\boldsymbol{g}$ is a NSG since $\boldsymbol{g}^{-\{1\}}$, $\boldsymbol{g}^{-\{1,2\}}$, and $\boldsymbol{g}^{-[\![1,3]\!]}$ are interlinked stars while $\boldsymbol{g}^{-[\![1,4]\!]}$ is an empty network.

In Figure 3, network $\boldsymbol{h}$ is such that $|h(i)| \geq |h(i+1)|$ for each $i \in [\![1, 6]\!]$. Network $\boldsymbol{h}$ is not a NSG since $\boldsymbol{h}^{-\{1\}}$ is neither an interlinked star, nor an empty network. However, network $\boldsymbol{h}$ is an interlinked star.

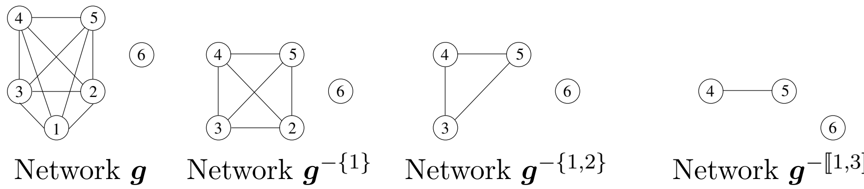


Figure 2: A NSG

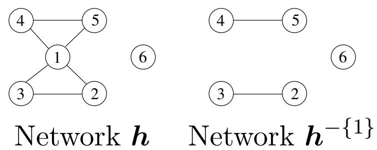


Figure 3: An interlinked star which is not a NSG

2.2 Order on networks

Let $Z(\boldsymbol{g}) = \{\{i,j\} \subset N \times N \mid g_{i,j} = 1\}$ be the set of pairs of players who are linked in $\boldsymbol{g}$. Let $\leq$ be the natural partial order relation on $\mathbb{N}^2$. This relation determines the set $\{((a,b),(a',b')) \in \mathbb{N}^2 : a \leq a', b \leq b'\}$. Since this set determines $\leq$, we might as well regard $\leq$ as being identical to this set. Let $\not\leq$ be the complement set of $\leq$ in $\mathbb{N}^2$.

We define two functions $\alpha_{\boldsymbol{g}} : Z(\boldsymbol{g}) \to [\![0, n-1]\!] \times [\![0, n-1]\!]$, $\alpha_{\boldsymbol{g}} : \{i,j\} \mapsto \alpha_{\boldsymbol{g}}(\{i,j\}) = (\max\{|g(i)|, |g(j)|\}, \min\{|g(i)|, |g(j)|\})$, and $\beta_{\boldsymbol{g}} : Z(\boldsymbol{g}) \to [\![0, n-1]\!] \times [\![0, n-1]\!]$, $\beta_{\boldsymbol{g}} : \{i,j\} \mapsto \beta_{\boldsymbol{g}}(\{i,j\}) = (\min\{|g(i)|, |g(j)|\}, \max\{|g(i)|, |g(j)|\})$. Let $I(\boldsymbol{g}) = \min_{\{i,j\} \in Z(\boldsymbol{g})}\{\alpha_{\boldsymbol{g}}(\{i,j\})\}$ and $J(\boldsymbol{g}) = \min_{\{i,j\} \in Z(\boldsymbol{g})}\{\beta_{\boldsymbol{g}}(\{i,j\})\}$, where min is defined with the natural order relation on $\mathbb{N}^2$. We now illustrate $I(\boldsymbol{g})$ and $J(\boldsymbol{g})$ through an example. For network $\boldsymbol{g}$ drawn in Figure 4, we have $\alpha_{\boldsymbol{g}}(\{1,k\}) = (4,2)$ and $\beta_{\boldsymbol{g}}(\{1,k\}) = (2,4)$ for all $k \in [\![2,5]\!]$, $\alpha_{\boldsymbol{g}}(\{2,3\}) = \alpha_{\boldsymbol{g}}(\{4,5\}) = \beta_{\boldsymbol{g}}(\{2,3\}) = \beta_{\boldsymbol{g}}(\{4,5\}) = (2,2)$. Consequently, $I(\boldsymbol{g}) = \{(2,2)\}$ and $J(\boldsymbol{g}) = \{(2,2)\}$.

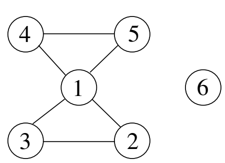


Figure 4: Networks $\boldsymbol{g}$

2.3 Payoffs

Let $\phi : [\![0, n-1]\!] \times [\![0, (n-2)(n-1)/2]\!] \to I\!R$, $\phi : (|g(i)|, L(\boldsymbol{g}^{-\{i\}})) \mapsto \phi(|g(i)|, L(\boldsymbol{g}^{-\{i\}}))$ be a function. We assume that $\phi(\cdot,\cdot)$ satisfies the following properties:

Property 1. $\phi(\cdot,\cdot)$ is convex in its first argument if for all $x \in [\![1, n-2]\!]$ and for all $y \in [\![0, (n-2)(n-1)/2]\!]$, $\phi(x+1, y) - \phi(x, y) \geq \phi(x, y) - \phi(x-1, y)$.

Property 2. $\phi(\cdot,\cdot)$ is strictly convex in its second argument if for all $x \in [\![0, n-1]\!]$ and for all $y \in [\![1, (n-2)(n-1)/2 - 1]\!]$, $\phi(x, y+1) - \phi(x, y) > \phi(x, y) - \phi(x, y-1)$.

Property 3. $\phi(\cdot,\cdot)$ is sub-modular if for all $x \in [\![0, n-2]\!]$, for all $y, y' \in [\![0, [(n-2)(n-1)/2]-1]\!]$ and $y \geq y'$, $\phi(x+1, y) - \phi(x, y) \leq \phi(x+1, y') - \phi(x, y')$.[8]

We consider network formation games with global spillovers, that is games in which the marginal returns from links for every player can be expressed in terms of the number of links of the player and the sum of links of the rest of the players. This class of games has been examined by Goyal and Joshi (2006). We define the payoff function of each player i as follows:

$$\pi_i(\boldsymbol{g}) = \phi\left(|g(i)|, L(\boldsymbol{g}^{-\{i\}})\right). \tag{1}$$

In the rest of the paper we are particularly interested in specific payoff functions. Let $\theta : (x, y) \mapsto (ax + by + c)^2 + f(x)$ be a function where $a, b, c \in I\!R$ and $f(\cdot)$ is a concave function. It is worth noting that if the quadratic form $(ax + by)^2$ satisfies Properties 1, 2 and 3, then function $\theta(\cdot,\cdot)$ satisfies Properties 1, 2 and 3. Moreover, the quadratic form $(ax+by)^2$ satisfies Properties 1, 2 and 3 if $\text{sign}(a) \neq \text{sign}(b)$. We now present an economic example to illustrate the choice of this payoff formulation and Properties 1, 2 and 3.

Example 1 *Cournot Oligopoly and Cost Reduction.* This example is taken from GJ (2003). Consider a homogeneous product Cournot oligopoly consisting of n *ex ante* identical firms which face the linear inverse demand function: $p = \alpha - \sum_{i \in N} q_i$, where p is the price of the product, q_i is the quantity produced by firm i, and $\alpha > 0$. The firms have zero fixed costs

[8] recall that this property is called strategic substitute property in Goyal and Joshi (2006).

and constant returns-to-scale cost functions. When firms enter in bilateral collaborations, they lower their marginal costs. More precisely, the marginal cost of firm i is given by $C(|g(i)|) = \gamma_0 - \gamma|g(i)|$, with $\gamma_0 > (n-1)\gamma > 0$. Given any network $\boldsymbol{g}$, the Cournot equilibrium output of firm $i \in N$ can be written as:

$$q_i(\boldsymbol{g}) = \frac{\alpha - \gamma_0 + (n-1)\gamma|g(i)| - 2\gamma L\left(\boldsymbol{g}^{-\{i\}}\right)}{n+1},$$

with $\alpha - \gamma_0 - (n-1)(n-2)\gamma > 0$ to ensure that each firm produces a strictly positive quantity in equilibrium. Let $f : |g(i)| \mapsto f(|g(i)|)$ be a function which measures the costs of forming links; we assume that $f(\cdot)$ is concave, that is for all $|g(i)| \in [\![2, n-2]\!]$, we have $f(|g(i)|+1) - f(|g(i)|) \leq f(|g(i)|) - f(|g(i)|-1)$. The Cournot profits for firm $i \in N$ are given by $\pi_i(\boldsymbol{g}) = (q_i(\boldsymbol{g}))^2 - f(|g(i)|)$. This means that $\pi_i(\boldsymbol{g}) = \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right))$. Since $\text{sign}(\gamma(n-1/n+1)) \neq \text{sign}(-2\gamma/n+1)$, $\phi(\cdot,\cdot)$ satisfies Properties 1, 2 , and 3. [9]

[9] It is worth noting that Properties 1, 2, and 3 are preserved in a differentiated product oligopoly consisting of n *ex ante* identical firms which compete either in quantities, or in prices. Suppose that each firm i faces the following linear inverse demand function: $p_i = \alpha - q_i - \beta \sum_{j \in N \setminus \{i\}} q_j$, where p_i is the price of the product sold by firm i, $\alpha > 0$, and $\beta \in]0, 1[$.

The Cournot equilibrium output of firm $i \in N$ is

$$q_i^C(\boldsymbol{g}) = \frac{(2-\beta)(\alpha - \gamma_0) + \gamma((n-3)\beta + 2)|g(i)| - 2\gamma\beta L(\boldsymbol{g}^{-\{i\}})}{(2-\beta)(2+\beta(n-1))}.$$

The Cournot profits for firm $i \in N$ are given by $\pi_i^C(\boldsymbol{g}) = \phi(|g(i)|, L(\boldsymbol{g}^{-\{i\}})) = (q_i^C(\boldsymbol{g}))^2 - f(|g(i)|)$. Since $\text{sign}(\gamma((n-3)\beta+2))((2-\beta)(2+\beta(n-1))) \neq \text{sign}(2\gamma\beta)((2-\beta)(2+\beta(n-1)))$, $\pi_i^C(\boldsymbol{g})$ satisfies Properties 1, 2, and 3.

In the Bertrand equilibrium, the profits for firm $i \in N$ can be written in the form: $\pi_i^B(\boldsymbol{g}) = \phi(|g(i)|, L(\boldsymbol{g}^{-\{i\}})) = (a|g(i)| + bL(\boldsymbol{g}^{-\{i\}}) + c)^2 - f(|g(i)|)$ where

$$a = \lambda^{\frac{1}{2}}\gamma \frac{2 + (5n-11)\beta + (4n^2 - 19n + 21)\beta^2 + ((n^2 - 8n + 19)n - 14)\beta^3}{(2 + (n-3)\beta)(1 + (n-1)\beta)(1-\beta)(2 + (2n-3)\beta)}$$

$$b = -\lambda^{\frac{1}{2}} 2\gamma \frac{\beta + (2n-4)\beta^2 + (n^2 - 4n + 4)\beta^3}{(2 + (n-3)\beta)(1 + (n-1)\beta)(1-\beta)(2 + (2n-3)\beta)}.$$

and

$$\lambda = \frac{(1-\beta)(1 + (n-1)\beta)}{1 + (n-2)\beta}.$$

Since $\text{sign}(a) \neq \text{sign}(b)$, $\pi_i^B(\boldsymbol{g})$ satisfies Properties 1, 2, and 3.

2.4 Efficient Networks

Let $W : \mathcal{G} \to I\!R$, $W : \boldsymbol{g} \mapsto W(\boldsymbol{g}) = \sum_{i \in N} \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)$ be a function. We call $W(\cdot)$ the total payoff function. Moreover, let $U : \mathcal{G} \to I\!R$, $U : \boldsymbol{g} \mapsto U(\boldsymbol{g})$ be another function. We say that $U(\cdot)$ is *architecture independent* if it satisfies the following property: If $\boldsymbol{g}$ and $\boldsymbol{g}'$ satisfy $L(\boldsymbol{g}) = L(\boldsymbol{g}')$, then $U(\boldsymbol{g}) = U(\boldsymbol{g}')$.

Consider a function $\psi : [\![0, n(n-1)/2]\!] \to I\!R$, $\psi : L(\boldsymbol{g}) \mapsto \psi(L(\boldsymbol{g}))$, and let $U(\boldsymbol{g}) = \psi(L(\boldsymbol{g}))$. Moreover, let $\psi(\cdot)$ be convex. In the following we will say that $U(\cdot)$ is architecture independent and convex when $\psi(\cdot)$ is convex. Finally, we define the global payoff function as follows: $SW : \mathcal{G} \to I\!R$, $SW : \boldsymbol{g} \mapsto W(\boldsymbol{g}) + U(\boldsymbol{g})$, where $W(\cdot)$ is the total payoff function and $U(\cdot)$ is architecture independent and convex.

Definition 4 *An efficient network $\boldsymbol{g}$ is a network which maximizes the sum of the players' payoffs. In other words, network $\boldsymbol{g} \in \mathcal{G}$ is efficient if $W(\boldsymbol{g}) \geq W(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{G}$.*

Definition 5 *A global efficient network $\boldsymbol{g}$ is a network which maximizes the function $SW(\boldsymbol{g})$.*

We now illustrate functions that are architecture independent and convex.

Example 2 *Cournot Oligopoly and Cost Reduction.* Suppose that the inverse demand function is given by: $p\left(\sum_{i \in N} q_i(\boldsymbol{g})\right) = \alpha - \sum_{i \in N} q_i(\boldsymbol{g})$. Then the consumer surplus in $\boldsymbol{g}$, $SC(\boldsymbol{g})$ is given by $1/2\left(\sum_{i \in n} q_i(\boldsymbol{g})\right)^2$. We have:

$$1/2\left(\sum_{i \in n} q_i(\boldsymbol{g})\right)^2 = \frac{1}{2}\left(\frac{n(\alpha - \gamma_0) + 2\gamma n L(\boldsymbol{g})}{n+1}\right)^2$$

For $\boldsymbol{g}$ and $\boldsymbol{g}'$, such that $L(\boldsymbol{g}) = L(\boldsymbol{g}')$, we have $SC(\boldsymbol{g}) = SC(\boldsymbol{g}')$, so $SC(\cdot)$ is architecture independent. We can write $SC(\boldsymbol{g}) = \psi(L(\boldsymbol{g}))$. Moreover we have $\psi(L(\boldsymbol{g}) + 1) - \psi(L(\boldsymbol{g})) > \psi(L(\boldsymbol{g})) - \psi(L(\boldsymbol{g}) - 1)$, so $\psi(\cdot)$ is convex.

3 Results

In this section, our main result establishes the architectures of efficient networks (Proposition 2) and global efficient networks (Corollary 3). But first we provide a proposition that allows us to check easily whether a network is candidate to be an efficient network. The proof of this proposition is given in the appendix.

Proposition 1 *Suppose that the payoff function is given by equation 1 and $\phi(\cdot,\cdot)$ satisfies Properties 1, 2 and 3. Then an efficient network $\boldsymbol{g}$ is such that for all $(|g(i)|,|g(i')|) \in I(\boldsymbol{g})$ and $(|\bar{g}(j)|,|\bar{g}(j')|) \in J(\bar{\boldsymbol{g}})$, we have $(|g(i)|+|\bar{g}(j)|,|g(i')|+|\bar{g}(j')|) \nleq (n-1,n-1)$.*

The intuition of the proof is as follows. Consider a network $\boldsymbol{g}$, with $g_{i,i'}=1$ and $g_{j,j'}=0$. Suppose $|g(i)| \leq |g(j)|$ and $|g(i')| \leq |g(j')|$. We draw networks $\boldsymbol{g}-ii'$, $\boldsymbol{g}$ and $\boldsymbol{g}+jj'$ in Figure 5 below, and we show that if the link ii' increases the total payoff, then the link jj' also increases the total payoff.

Consider players in $N \setminus \{i,i',j,j'\}$. For each player $k \in N \setminus \{i,i',j,j'\}$, by Property 2, the marginal payoff she obtains when the link jj' is added to $\boldsymbol{g}$ is strictly higher than the marginal payoff she obtains when the link ii' is added to $\boldsymbol{g}-ii'$.

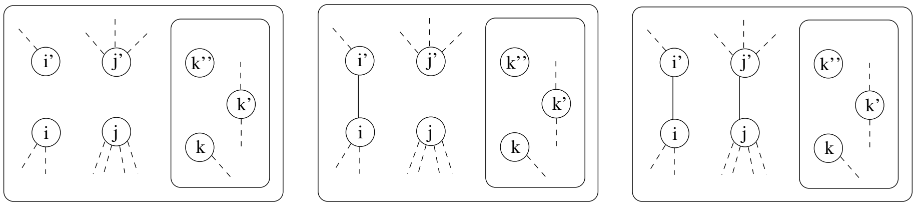
Consider players i and j. First, we show that the marginal payoff obtained by j when the link jj' is added to $\boldsymbol{g}$, $\Delta\pi'_j$, is higher than the marginal payoff obtained by i when the link ii' is added to $\boldsymbol{g}-ii'$, $\Delta\pi_i$. We note that i and j increase their number of links by one, j has formed more links in $\boldsymbol{g}$ than i in $\boldsymbol{g}-ii'$, and j faces less links in $\boldsymbol{g}$ than i in $\boldsymbol{g}-ii'$. Properties 1 and 3, we conclude that $\Delta\pi'_j \geq \Delta\pi_i$.

Second, we show that the marginal payoff obtained by i when the link jj' is added to $\boldsymbol{g}$, $\Delta\pi'_i$, is higher than the marginal payoff obtained by j when the link ii' is added to $\boldsymbol{g}-ii'$, $\Delta\pi_j$. We note that i and j face players who increase their number of links by one, j has formed more links in $\boldsymbol{g}$ than i in $\boldsymbol{g}-ii'$, and j faces less links in $\boldsymbol{g}$ than i in $\boldsymbol{g}-ii'$. By Properties 2 and 3, we conclude that $\Delta\pi'_i \geq \Delta\pi_j$.

It follows that the sum of the marginal payoffs of players i and j is higher when the link jj' is added to $\boldsymbol{g}$ than when the link ii' is added to $\boldsymbol{g}-ii'$.

Consider players i' and j'. With similar arguments as for players i and j, we establish that the sum of the marginal payoffs of i' and j' is higher when the link jj' is added than when the link ii' is added.

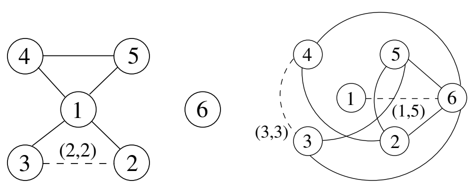
We can conclude that if the link ii' increases the total payoff, then the link jj' also increases the total payoff.



Network $\boldsymbol{g} - ii'$ Network $\boldsymbol{g}$ Network $\boldsymbol{g} + jj'$

Figure 5: Intuition of Proposition 1

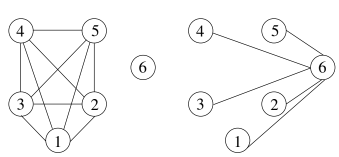
We can deduce a method from Proposition 1 which allows us to know whether a network is candidate to be an efficient network. We illustrate this method through the following example. Let $N = [\![1, 6]\!]$. First we deal with network $\boldsymbol{g}$ drawn in Figure 6. We build the complement network of $\boldsymbol{g}$, $\bar{\boldsymbol{g}}$. We now determine $I(\boldsymbol{g})$ and $J(\bar{\boldsymbol{g}})$, we have $I(\boldsymbol{g}) = \{(2,2)\}$ and $J(\bar{\boldsymbol{g}}) = \{(3,3),(1,5)\}$. We have $(3,3) + (2,2) = (5,5) \leq (5,5)$. By Proposition 1, network $\boldsymbol{g}$ is not candidate to be efficient.



Network $\boldsymbol{g}$ Network $\bar{\boldsymbol{g}}$

Figure 6: Networks $\boldsymbol{g}$ and $\bar{\boldsymbol{g}}$

Second, we consider the network $\boldsymbol{h}$ drawn in Figure 7. We build the complement network of $\boldsymbol{h}$, $\bar{\boldsymbol{h}}$. We now determine $I(\boldsymbol{h})$ and $J(\bar{\boldsymbol{h}})$, we have $I(\boldsymbol{h}) = \{(4,4)\}$ and $J(\bar{\boldsymbol{h}}) = \{(1,5)\}$. We have $(4,4) + (1,5) = (5,9) \not\leq (5,5)$. By Proposition 1, network $\boldsymbol{h}$ is a candidate for being an efficient network.



Network $\boldsymbol{h}$ Network $\bar{\boldsymbol{h}}$

Figure 7: Networks $\boldsymbol{h}$ and $\bar{\boldsymbol{h}}$

It is worth noting that the above procedure consists in finding a minimum element from a list of pairs of degrees. It is well known that there exist polynomial time algorithms to solve this kind of problems.

Corollary 1 *Suppose that the payoff function is given by equation 1 and $\phi(\cdot,\cdot)$ satisfies Properties 1, 2 and 3, and $U(\cdot)$ is architecture independent and convex. Then a global efficient network $\boldsymbol{g}$ is such that for all $(|g(i)|,|g(i')|) \in I(\boldsymbol{g})$ and $(|\bar{g}(j)|,|\bar{g}(j')|) \in J(\bar{\boldsymbol{g}})$, we have $(|g(i)|+|\bar{g}(j)|,|g(i')|+|\bar{g}(j')|) \not\leq (n-1,n-1)$.*

Proof The proof is based on the same arguments as the proof of Proposition 1 and the fact that $U(\boldsymbol{g})$ is architecture independent and convex. Under the assumptions given in the proof of Proposition 1 we have $W(\boldsymbol{g}+jj')-W(\boldsymbol{g}) > W(\boldsymbol{g})-W(\boldsymbol{g}-ii')$. Moreover, since $\psi(\cdot)$ is convex, we have $U(\boldsymbol{g}+jj')-U(\boldsymbol{g}) \geq U(\boldsymbol{g})-U(\boldsymbol{g}-ii')$. We obtain $SW(\boldsymbol{g}+jj')-SW(\boldsymbol{g}) > SW(\boldsymbol{g})-SW(\boldsymbol{g}-ii') \geq 0$. The first inequality follows the addition of functions $W(\cdot)$ and $U(\cdot)$. The second inequality comes from the fact that $\boldsymbol{g}$ is an efficient network. We obtain the desired contradiction. □

Corollary 2 *Suppose that the payoff function is given by equation 1 and $\phi(\cdot,\cdot)$ satisfies Properties 1, 2 and 3. Then in an efficient network the players who are involved in the highest number of links are connected with each non isolated player.*

Proof Let $\boldsymbol{g}$ be an efficient network and let i be a player who is involved in the highest number of links in $\boldsymbol{g}$. To introduce a contradiction suppose that there is a player j such that $|g(j)| \geq 1$ and $g_{i,j}=0$. Since $|g(i)| \geq |g(j)|$, we have $|\bar{g}(i)| \leq |\bar{g}(j)|$. Since i is involved in the highest number of links and $|g(j)| \geq 1$, there exists a player, say k, such that $g_{k,j}=1$. In the following, we assume that $(\max\{|g(j)|,|g(k)|\}, \min\{|g(j)|,|g(k)|\}) \in I(\boldsymbol{g})$ and $(|\bar{g}(i)|,|\bar{g}(j)|) \in J(\bar{\boldsymbol{g}})$.[10] There are two possibilities.

1. Suppose $|g(j)| \geq |g(k)|$. Then $(|g(j)|,|g(k)|) \in I(\boldsymbol{g})$. We have $(|g(j)|,|g(k)|)+(|\bar{g}(i)|,|\bar{g}(j)|) = (|g(j)|+|\bar{g}(i)|,|g(k)|+|\bar{g}(j)|) = (n-1+|g(j)|-|g(i)|,n-1+|g(k)|-|g(j)|)$. Consequently, $(|g(j)|+|\bar{g}(i)|,|g(k)|+|\bar{g}(j)|) \leq (n-1,n-1)$ since $n-1+|g(j)|)-|g(i)| \leq n-1$ and $n-1+|g(k)|)-|g(j)| \leq n-1$, a contradiction by Proposition 1.

[10] If it is not the case first we find two pairs $(|g(j')|,|g(k')|) \in I(\boldsymbol{g})$ and $(|\bar{g}(k'')|,|\bar{g}(j'')|) \in J(\bar{\boldsymbol{g}})$ which satisfy $(|g(j')|,|g(k')|) \leq (\max\{|g(j)|,|g(k)|\}, \min\{|g(j)|,|g(k)|\})$ and $(|\bar{g}(k'')|,|\bar{g}(j'')|) \leq (|\bar{g}(i)|,|\bar{g}(j)|)$. Then we apply the arguments given above on these two pairs.

2. Suppose $|g(j)| < |g(k)|$. It follows that $(|g(k)|, |g(j)|) \in I(\boldsymbol{g})$. We have $(|g(k)|, |g(j)|) + (|\bar{g}(i)|, |\bar{g}(j)|) = (|g(k)| + |\bar{g}(i)|, |g(j)| + |\bar{g}(j)|) = (n-1+|g(k)|-|g(i)|, n-1)$. Consequently, $(|g(k)|+|\bar{g}(i)|, |g(j)|+|\bar{g}(j)|) \leq (n-1, n-1)$, since $n-1+|g(k)|-|g(i)| \leq n-1$, a contradiction by Proposition 1. □

It is worth noting that a network $\boldsymbol{g}$, where the players who are involved in the highest number of links are connected with each non isolated player, is an interlinked star. We now present the main result of the paper.

Proposition 2 *Suppose that the payoff function is given by equation 1 and $\phi(\cdot,\cdot)$ satisfies Properties 1, 2 and 3. Then an efficient network is a NSG.*

Proof By Corollary 1, we know that an efficient network $\boldsymbol{g}$ is an interlinked star. We assume that $|g(1)| \geq |g(2)| \geq \ldots \geq |g(n)|$. We know that player 1 is involved in a link with each non isolated player i in $\boldsymbol{g}$. It follows that for all players $i \in N$ such that $|g^{-\{1\}}(i)| \geq 1$, we have $|g^{-\{1\}}(i)| + 1 = |g(i)|$. If $|g^{-\{1\}}(i)| = 0$ for all $i \in N$, then we are done. Otherwise we show that $\boldsymbol{g}^{-\{1\}}$ is an interlinked star. To introduce a contradiction, suppose that there exists a player $j \in N \setminus \{2\}$ such that $g_{2,j} = 0$ and $|g^{-\{1\}}(j)| \geq 1$. Let k be a player such that $g^{-\{1\}}_{j,k} = 1$. There are two possibilities.

1. Suppose that $|g^{-\{1\}}(j)| \geq |g^{-\{1\}}(k)|$. Without loss of generality, we suppose that $(|g^{-\{1\}}(j)|, |g^{-\{1\}}(k)|) \in I(\boldsymbol{g}^{-\{1\}})$ and $(|\bar{g}^{-\{1\}}(2)|, |\bar{g}^{-\{1\}}(j)|) \in J(\bar{\boldsymbol{g}}^{-\{1\}})$. We have $(|g^{-\{1\}}(j)|+|\bar{g}^{-\{1\}}(2)|, |g^{-\{1\}}(k)| + |\bar{g}^{-\{1\}}(j)|) = (n-1+|g^{-\{1\}}(j)| - |g^{-\{1\}}(2)|,\ n-1+|g^{-\{1\}}(k)| - |g^{-\{1\}}(j)|) = (n-1+|g(j)|-|g(2)|, n-1+|g(k)|-|g(j)|) \leq (n-1, n-1)$ since $n-1+|g(j)|-|g(2)| \leq n-1$ and $n-1+|g(k)|-|g(j)| \leq n-1$. A contradiction by Proposition 1.

2. Suppose that $|g^{-\{1\}}(j)| < |g^{-\{1\}}(k)|$, without loss of generality, we suppose that $(|g^{-\{1\}}(k)|, |g^{-\{1\}}(j)|) \in I(\boldsymbol{g}^{-\{1\}})$ and $(|\bar{g}^{-\{1\}}(2)|, |\bar{g}^{-\{1\}}(j)|) \in J(\bar{\boldsymbol{g}}^{-\{1\}})$. We have $(|g^{-\{1\}}(k)|+|\bar{g}^{-\{1\}}(2)|, |g^{-\{1\}}(j)|+|\bar{g}^{-\{1\}}(j)|) = (n-1+|g^{-\{1\}}(k)|-|g^{-\{1\}}(2)|, n-1) = (n-1+|g(k)|-|g(2)|, n-1) \leq (n-1, n-1)$ since $n-1+|g(k)|-|g(2)| \leq n-1$. A contradiction by Proposition 1.

We reiterate these arguments for each network $\boldsymbol{g}^{-X}$, where $X = [\![1,3]\!], [\![1,4]\!], \ldots, [\![1,\ell]\!]$ for each $\ell \in [\![3, n-1]\!]$. There exists $\ell^\star$ such that $\boldsymbol{g}^{-[\![1,\ell]\!]}$ is an interlinked star for all $\ell \leq \ell^\star$ while $\boldsymbol{g}^{-[\![1,\ell]\!]}$ is empty for all $\ell > \ell^\star$ is empty. □

Example 3 *Cournot Oligopoly and Cost Reduction.* By Proposition 1, a network which maximizes the total profit of firms is a NSG.

By using similar arguments to those given in the proof of Proposition 1, we obtain the following result.

Corollary 3 *Suppose that the payoff function is given by equation 1 and $\phi(\cdot,\cdot)$ satisfies Properties 1, 2 and 3, and $U(\cdot)$ is architecture independent and convex. Then a global efficient network is a NSG.*

Example 4 *Cournot Oligopoly and Cost Reduction.* By Corollary 3, a network which maximizes the social welfare is a NSG.

It is worth noting that the class of networks we identify as candidates for efficient networks is smaller than the class provided by Westbrock (Proposition 1, pg. 358, 2010). Indeed, network $\boldsymbol{g}$ in Figure 4 is candidate to be an efficient network according to Westbrock while it is not the case in our paper since $\boldsymbol{g}$ is not a NSG.

4 Conclusion

In this paper, we have added to the work of Westbrock (2010) by characterizing architectures of efficient networks in a more general class of games than those examined by him. Additionally, we have shown that networks which are candidates to be efficient are NSG and have related this class of networks to interlinked stars, which are the architectures used by Westbrock.
Moreover, we have also completed the work of Goyal and Joshi (2006) by characterizing efficient networks for network formation games under global spillovers with convexity and submodularity. The characterization of these efficient networks comes at a small cost - unlike GJ we need to assume that the payoff function of each player i is strictly convex in the number of links in which i is not involved.

5 Appendix

Proof of Proposition 1. Let $\boldsymbol{g}$ be an efficient network. To introduce a contradiction, suppose that there exist two pairs, say $(i, i') \in N \times N$ and $(j, j') \in N \times N$, in $\boldsymbol{g}$ such that

$(|g(i)|,|g(i')|) \in I(\boldsymbol{g})$, $(|\bar{g}(j)|,|\bar{g}(j')|) \in J(\bar{\boldsymbol{g}})$ and $(|g(i)|+|\bar{g}(j)|,|g(i')|+|\bar{g}(j')|) \leq (n-1,n-1)$. Since $(|g(i)|,|g(i')|) \in I(\boldsymbol{g})$ and $(|\bar{g}(j)|,|\bar{g}(j')|) \in J(\bar{\boldsymbol{g}})$, we have $g_{i,i'} = 1$ and $g_{j,j'} = 0$. Moreover, since $(|g(i)|+|\bar{g}(j)|,|g(i')|+|\bar{g}(j')|) \leq (n-1,n-1)$, we have $|g(i)| \leq n-1-|\bar{g}(j)| = |g(j)|$ and $|g(i')| \leq n-1-|\bar{g}(j')| = |g(j')|$. Finally, there are $|L(\boldsymbol{g})|$ links in $\boldsymbol{g}$. Hence, if $k \in N$ is involved in $|g(k)|$ links, then there are $L\left(\boldsymbol{g}^{\{-k\}}\right) = |L(\boldsymbol{g})| - |g(k)|$ links in which k is not involved. It follows that if $|g(i)| \leq |g(j)|$, then $L\left(\boldsymbol{g}^{-\{i\}}\right) \geq L\left(\boldsymbol{g}^{\{-j\}}\right)$ in $\boldsymbol{g}$. In the following, we denote by $N' = N \setminus \{i,j,i',j'\}$. Let us compare $W(\boldsymbol{g}+jj')$ and $W(\boldsymbol{g})$:

$$\begin{aligned}
W(\boldsymbol{g}+jj') - W(\boldsymbol{g}) \quad = \quad & \phi(|g(j)|+1, L\left(\boldsymbol{g}^{\{-j\}}\right)) - \phi(|g(j)|, L\left(\boldsymbol{g}^{\{-j\}}\right)) \\
& +\phi(|g(j')|+1, L\left(\boldsymbol{g}^{\{-j'\}}\right)) - \phi(|g(j')|, L\left(\boldsymbol{g}^{\{-j'\}}\right)) \\
& +\textstyle\sum_{\ell \in N'} [\phi(|g(\ell)|, L\left(\boldsymbol{g}^{\{-\ell\}}\right)+1) - \phi(|g(\ell)|, L\left(\boldsymbol{g}^{\{-\ell\}}\right))] \\
& +\phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)+1) - \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)) \\
& +\phi(|g(i')|, L\left(\boldsymbol{g}^{\{-i'\}}\right)+1) - \phi(|g(i')|, L\left(\boldsymbol{g}^{\{-i'\}}\right)).
\end{aligned}$$

We now compare $W(\boldsymbol{g})$ and $W(\boldsymbol{g}-ii')$.

$$\begin{aligned}
W(\boldsymbol{g}) - W(\boldsymbol{g}-ii') \quad = \quad & \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)) - \phi(|g(i)|-1, L\left(\boldsymbol{g}^{-\{i\}}\right)) \\
& +\phi(|g(i')|, L\left(\boldsymbol{g}^{\{-i'\}}\right)) - \phi(|g(i')|-1, L\left(\boldsymbol{g}^{\{-i'\}}\right)) \\
& +\textstyle\sum_{\ell \in N'} [\phi(|g(\ell)|, L\left(\boldsymbol{g}^{\{-\ell\}}\right)) - \phi(|g(\ell)|, L(\boldsymbol{g}^{\{-\ell\}})-1)] \\
& +\phi(|g(j)|, L(\boldsymbol{g}^{\{-j\}})) - \phi(|g(j)|, L(\boldsymbol{g}^{\{-j\}})-1) \\
& +\phi(|g(j')|, L(\boldsymbol{g}^{\{-j'\}})) - \phi(|g(j')|, L(\boldsymbol{g}^{\{-j'\}})-1).
\end{aligned}$$

Since $\boldsymbol{g}$ is efficient, we have $W(\boldsymbol{g}) - W(\boldsymbol{g}-ii') \geq 0$ and $W(\boldsymbol{g}+jj') - W(\boldsymbol{g}) \leq 0$. To obtain a contradiction we have to show that $W(\boldsymbol{g}+jj') - W(\boldsymbol{g}) > W(\boldsymbol{g}) - W(\boldsymbol{g}-ii')$.

To conclude, we deal with five quantities $\mathcal{A}_1(\boldsymbol{g})$, $\mathcal{A}_2(\boldsymbol{g})$, $\mathcal{A}_3(\boldsymbol{g})$, $\mathcal{A}_4(\boldsymbol{g})$, $\mathcal{A}_5(\boldsymbol{g})$.

First, we deal with

$$\begin{aligned}\mathcal{A}_1(\boldsymbol{g}) &= \textstyle\sum_{\ell\in N'}([\phi(|g(\ell)|, L(\boldsymbol{g}^{\{-\ell\}})+1) - \phi(|g(\ell)|, L(\boldsymbol{g}^{\{-\ell\}}))] \\ &\quad - \textstyle\sum_{\ell\in N'}[\phi(|g(\ell)|, L(\boldsymbol{g}^{\{-\ell\}})) - \phi(|g(\ell)|, L(\boldsymbol{g}^{\{-\ell\}})-1)]\end{aligned}$$

We have $\mathcal{A}_1(\boldsymbol{g}) > 0$ since $\phi(\cdot,\cdot)$ is strictly convex in its second argument.

We now deal with

$$\begin{aligned}\mathcal{A}_2(\boldsymbol{g}) &= \phi(|g(j)|+1, L(\boldsymbol{g}^{\{-j\}})) - \phi(|g(j)|, L(\boldsymbol{g}^{\{-j\}})) \\ &\quad -[\phi(|g(i)|, L(\boldsymbol{g}^{\{-i\}})) - \phi(|g(i)|-1, L(\boldsymbol{g}^{-\{i\}}))].\end{aligned}$$

We have $\mathcal{A}_2(\boldsymbol{g}) \geq 0$ since

$$\begin{aligned}\phi(|g(j)|+1, L(\boldsymbol{g}^{\{-j\}})) - \phi(|g(j)|, L(\boldsymbol{g}^{\{-j\}})) &\geq \phi(|g(j)|+1, L\left(\boldsymbol{g}^{-\{i\}}\right)) - \phi(|g(j)|, L\left(\boldsymbol{g}^{-\{i\}}\right)) \\ &\geq \phi(|g(i)|+1, L\left(\boldsymbol{g}^{-\{i\}}\right)) - \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)) \\ &\geq \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)) - \phi(|g(i)|-1, L\left(\boldsymbol{g}^{-\{i\}}\right)).\end{aligned}$$

The first inequality comes from the sub-modularity of $\phi(\cdot,\cdot)$. The second and the third inequalities come from the convexity of $\phi(\cdot,\cdot)$ in its first argument.

We use the same arguments to show that

$$\begin{aligned}\mathcal{A}_3(\boldsymbol{g}) &= \phi(|g(j')|+1, L(\boldsymbol{g}^{\{-j'\}})) - \phi(|g(j')|, L(\boldsymbol{g}^{\{-j'\}})) \\ &\quad -[\phi(|g(i')|, L(\boldsymbol{g}^{\{-i'\}})) - \phi(|g(i')|-1, L(\boldsymbol{g}^{\{-i'\}}))] \\ &\geq 0.\end{aligned}$$

We now deal with

$$\begin{aligned}\mathcal{A}_4(\boldsymbol{g}) &= \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)+1) - \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)) \\ &\quad -[\phi(|g(j)|, L\left(\boldsymbol{g}^{\{-j\}}\right)) - \phi(|g(j)|, L\left(\boldsymbol{g}^{\{-j\}}\right)-1)].\end{aligned}$$

We have $\mathcal{A}_4(\boldsymbol{g}) > 0$ since

$$\begin{aligned}\phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)+1) - \phi(|g(i)|, L\left(\boldsymbol{g}^{-\{i\}}\right)) &> \phi(|g(i)|, L\left(\boldsymbol{g}^{\{-j\}}\right)) - \phi(|g(i)|, L\left(\boldsymbol{g}^{\{-j\}}\right)-1) \\ &\geq \phi(|g(j)|, L\left(\boldsymbol{g}^{\{-j\}}\right)) - \phi(|g(j)|, L\left(\boldsymbol{g}^{\{-j\}}\right)-1).\end{aligned}$$

The first inequality comes from the convexity of $\phi(\cdot,\cdot)$ in its second argument. The second inequality comes from the submodularity of $\phi(\cdot,\cdot)$.

We use the same arguments to show that

$$\begin{aligned}\mathcal{A}_5(\boldsymbol{g}) &= \phi(|g(i')|, L\left(\boldsymbol{g}^{\{-i'\}}\right)+1) - \phi(|g(i')|, L\left(\boldsymbol{g}^{\{-i'\}}\right)) \\ &\quad -[\phi(|g(j')|, L\left(\boldsymbol{g}^{\{-j'\}}\right)) - \phi(|g(j')|, L\left(\boldsymbol{g}^{\{-j'\}}\right)-1)]. \\ &> 0.\end{aligned}$$

To sum up, we have:

$$W(\boldsymbol{g}+jj') - W(\boldsymbol{g}) - [W(\boldsymbol{g}) - W(\boldsymbol{g}-ii')] = \sum_{\ell=1}^{5} A_\ell(\boldsymbol{g}) > 0,$$

since $A_\ell(\boldsymbol{g}) \geq 0$, for $\ell \in [\![1,5]\!]$, and $A_1(\boldsymbol{g}), A_4(\boldsymbol{g}), A_5(\boldsymbol{g}) > 0$. It follows that $W(\boldsymbol{g}+jj') - W(\boldsymbol{g}) > W(\boldsymbol{g}) - W(\boldsymbol{g}-ii') \geq 0$ and $\boldsymbol{g}$ is not an efficient network. □

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